Model Order Reduction for Discrete-Time LTI Systems Using the Alignment Distance

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***Abstract*— The alignment distance is a recently introduced (differential-geometric) distance on the manifold of LTI systems of fixed order n and output-input dimension (p, m). In this paper, we formulate model order reduction for discrete-time LTI (MIMO) systems in terms of the alignment distance. The intuition behind our formulation is to consider systems of orders lower than n as boundary points of the mentioned manifold in an appropriate ambient space, and the goal is to find a system of order at most r (on the boundary) closest to a given system of order n, where closeness is measured in the alignment distance. We introduce an algorithm for this minimization problem and give some a-priori error bounds in terms of the Hankel singular values of the system. Interesting relations and resemblances emerge with the popular balanced truncation reduction, which is a method not based on any optimality criterion. We show that in certain cases (but not always) balanced truncation provides a good approximation to reduction based on the alignment distance. In fact, our approach can be considered as a principled attempt to put balanced truncation in an optimization framework, and in doing so we allude to a shortcoming of balanced truncation that highlights an advantage of our approach. The proposed approach is general and can be extended to other classes of systems.**

I. Introduction

Model order reduction is an important and old problem in control theory with applications in the analysis, design, and implementation of systems and controllers. Despite its long history and great importance, the definite solution to it has proven to be elusive. This is partly due to the fact that the problem itself is not quite well defined. In its simplest form the problem can be expressed as follows: Given a linear time-invariant (LTI) system of order n find another system of order $r < n$ that best approximates the system in "an appropriate sense." An important challenge is that the appropriate sense highly depends on the application or even the designer.

An ideal model order reduction method should have a rather long list of desirable (often contradicting) properties. For example, it should be theoretically sound yet conceptually simple, have nice control-theoretic features, be computationally friendly, and work well in practice, i.e., be effective in achieving the good reduction the designer expects. Certain approaches are quite natural, e.g., the L^2 model order reduction where the closest (in the L^2-norm between impulse responses) system of order r is sought [28], [17]. However, this approach results in a highly nonlinear non-convex optimization. Another issue with this approach is that it is only applicable to stable systems, whereas one often wants to reduce the order of unstable systems in a way that any design (e.g., a stabilizing controller) applied to the reduced-order system works well for the original system. In fact, a desirable control-theoretic property is robustness, especially robustness of stability under feedback (see [10], [19] on how the sense of approximation can relate to robustness of stability under feedback). In that sense the *gap* metric would be an ideal candidate distance to formulate model order reduction in terms of [8]. However, model reduction in terms of the gap metric again results in highly complicated algorithms (see [11], [24] and references therein).

Among existing algorithms, Moore's *diagonally*[1] *balanced* truncation [23] seems to be the most popular algorithm in practice. This is presumably mainly due to its simple and intuitive framework, ease of implementation, and good or acceptable practical results. However, this approach also lacks a sound theoretical foundation (see e.g., Section V in [23] for the author's view on "theoretical gaps" in the approach), is not applicable to unstable and stable systems seamlessly (although some workarounds have been proposed [22]), and more importantly has some shortcomings in capturing the dynamics of the model (this issue is really linked to the first criticism, see [20] and our discussion in §VI). Overall, an all-around solution to model reduction remains highly elusive. The reader is referred to [26] and [4] for various aspects of the problem of model reduction.

In this paper we try to address model order reduction from a new point of view with the hope of addressing some of the shortcomings in this area. The proposed approach is based on a recently introduced distance on the manifold of LTI systems of fixed order and size called the *alignment* distance [2]. The core idea behind this new distance is to find the change of basis that best aligns the state-space realizations of two given LTI systems and then compare the aligned realizations to get a distance between the systems (see Figure 1). Its derivation is based on some basic notions from differential geometry. The setting in which the alignment distance can be defined is quite general, e.g., deterministic, stochastic, discrete or continuous-time systems can be considered (see [2], [3], [1]). So far we have used this distance mainly for applications in comparing linear dynamical systems in modeling high-dimensional time-series data such as video

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[1] In the paper we will be dealing with a more general form of realization balancing where the controllability and observability Gramians are equal but not necessarily diagonal; hence we distinguish between balancing and diagonal balancing (see §III-A)

sequences [1]. The goal (and hope) of this paper is to show that the alignment distance can be useful in solving important control problems, e.g., model order reduction can be expressed naturally in the alignment distance.

When compared with existing approaches, our approach has multiple advantages, it has a sound theoretical foundation, it can be applied to unstable and stable systems as well as to stochastic systems, it is especially suited for Multi-Input Multi-Output (MIMO) systems, and it is conceptually simple and natural.[2] Also, as we show, it can alleviate a shortcoming of the popular diagonally balanced truncation method. Nevertheless, the computational aspect of our approach is not ideal (mainly the optimization problem involved is not convex, thus a-priori it might have local solutions and our algorithm may converge to such a solution). We should recall, however, that many model order reduction methods based on optimization also deal with non-convex optimization and among them only the Hankel norm method remarkably finds in closed-form all the (globally) optimal solutions [12]. We stress that the alignment distance and our approach in model reduction differ substantially from most traditional "approximation senses" and existing approaches; however, as mentioned before, the introduced model reduction method is closely related to the balanced truncation algorithm.

The remainder of this paper is organized as follows. In §II we recall some basic definitions and establish notations. In §III we give some preliminaries about the alignment distance. We try to avoid differential-geometric discussions and rely on intuition. In §IV we formulate the model reduction problem in the sense of the alignment distance. In §V we give an algorithm for solving the model reduction problem. In §VI we give an a-priori error bound on the approximation error (which resembles the well-known bound of Enns [9]). We also discuss connections with the diagonally balanced truncation method and a shortcoming of this method, which our approach alleviates. In §VII a typical model reduction problem is solved as an example and finally §VIII concludes the paper. Due to space limitation some more theoretical aspects of the results (including some of the proofs) and more discussions will not appear in this paper.

II. Basics and Notation

We consider a deterministic discrete-time LTI system M of *order* n and *size* (p, m) (i.e., m-dimensional input and p-dimensional output) described by:

$$\begin{aligned} x_t &= Ax_{t-1} + Bu_t \\ y_t &= Cx_t, \end{aligned} \tag{1}$$

where $R = (A, B, C) \in \widetilde{\mathcal{L}}_{m,n,p} = \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n}$ is called *a realization* of M. Here, u_t is the m-dimensional input assumed to be a *deterministic* signal stimulus. Given a positive integer $r < n$, partition the matrices conformably as $A = \left[\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} B_{11} \\ B_{21} \end{smallmatrix}\right]$ $C = \left[\begin{smallmatrix} C_{11} & C_{12} \end{smallmatrix}\right]$, where A_{11} is $r \times r$. We call $R_{11} = (A_{11}, B_{11}, C_{11})$ and $R_{22} = (A_{22}, B_{21}, C_{12})$ the top (or 11) and the bottom (or 22) sub-realizations of R, respectively. For a (possibly unstable) realization denote by $\mathcal{O}_k = [C^\top, (CA)^\top, \ldots, (CA^{k-1})^\top]^\top$ and $\mathcal{C}_k = [B, AB, \ldots, A^{k-1}B]$ the observability and controllability matrices of order k ($n \le k \le \infty$). Here $^\top$ denotes the transpose operation. The observability and controllability Gramians of order k are defined as $W_{\mathrm{o},k} = \mathcal{O}_k^\top \mathcal{O}_k$ and $W_{\mathrm{c},k} = \mathcal{C}_k \mathcal{C}_k^\top$, respectively. For $k = \infty$ (in which case we write simply W_o and W_c) asymptotic stability of A is needed. For an asymptotically stable realization R the controllability and observability Gramians satisfy the Lyapunov equations

$$W_c = BB^\top + AW_cA^\top, \tag{2a}$$

$$W_o = C^\top C + A^\top W_o A. \tag{2b}$$

The Hankel singular values of the system M (which are invariant under the state-space change of coordinate) are the square roots of the eigenvalues of W_oW_c, which we denote by $\lambda_1 \ge \lambda_2 \ldots \ge \lambda_n$. We denote the diagonal matrix comprised of these singular values as Λ. If the realization R is diagonally balanced [23] then we can assume $W_c = W_o = \Lambda$. Given a diagonally balanced realization R, we always assume that the singular values are decreasingly ordered and we call R_{11} and R_{22} the strong and weak sub-realizations, respectively (see [23] for reasons for this naming).

III. Preliminaries About the Alignment Distance

The reader is referred to [2] for a detailed and rigorous introduction to the alignment distance. Here, we rely more on intuition. We explicitly distinguish between a realization R and M, the system realized by R, which has an equivalent class of realizations all related by a change of coordinates under the *group* of nonsingular $n \times n$ matrices denoted by $GL(n)$. Specifically, let us denote the change of coordinates or the so-called similarity *action* by $\circ$, where for any $P \in GL(n)$

$$P \circ R = (P^{-1}AP, P^{-1}B, CP). \tag{3}$$

Then R and $P \circ R$ realize the same system from an input-output point of view. In fact, each LTI system of the form (1) has an equivalent class of realizations. In other words the space of systems can be considered as the *quotient* space of the space realizations under the above action of the group $GL(n)$. We write $\mathcal{L}_{m,n,p} = \widetilde{\mathcal{L}}_{m,n,p}/GL(n)$.

It turns out that the space $\mathcal{L}_{m,n,p}$ is not a nice mathematical objet (e.g., it is not a smooth manifold). However, if we restrict attention to the space (manifold) of minimal realizations $\widetilde{\Sigma}^{\min}_{m,n,p}$ or asymptotically stable minimal realizations $\widetilde{\Sigma}^{\min,\mathrm{a}}_{m,n,p}$, then their respective quotient spaces (namely $\Sigma^{\min}_{m,n,p} \triangleq \widetilde{\Sigma}^{\min}_{m,n,p}/GL(n)$ and $\Sigma^{\min,\mathrm{a}}_{m,n,p} \triangleq \widetilde{\Sigma}^{\min,\mathrm{a}}_{m,n,p}/GL(n)$) are smooth manifolds of dimension $n(m+p)$. Here smoothness comes from the usual notion of smoothness in the Euclidean space $\widetilde{\mathcal{L}}_{m,n,p}$. The realization-space pairs $(\widetilde{\Sigma}^{\min}_{m,n,p}, \Sigma^{\min}_{m,n,p})$ and $(\widetilde{\Sigma}^{\min,\mathrm{a}}_{m,n,p}, \Sigma^{\min,\mathrm{a}}_{m,n,p})$ form an object called principal fiber

[2] To be precise, in this paper we will not study all the enumerated aspects. However, we mention that the formulation in IV can be applied to both stable and unstable systems but the algorithm presented in §V is solely applicable to asymptotically stable systems. In a later work the other case will be addressed. Similarly the alignment distance for stochastic systems has been introduced [3] and model order reduction in this case can be pursued along the same lines presented here; however, the details will appear elsewhere.

bundle with structure group $GL(n)$. Defining a (group action induced) distance on the bottom or base space Σ of a generic principal bundle $(\widetilde{\Sigma}, \Sigma)$ with structure group $GL(n)$ is conceptually simple: Given a $GL(n)$-invariant distance $d_{\widetilde{\Sigma}}$ on the top space one defines $d_{\Sigma}(M_1, M_2) = \inf_{P \in GL(n)} d_{\widetilde{\Sigma}}(P \circ R_1, R_2)$, where $R_i (i = 1, 2)$ is *any* realization (or representation) of M_i. The distance $d_{\widetilde{\Sigma}}(P \circ R_1, R_2)$ is $GL(n)$-invariant if $d_{\widetilde{\Sigma}}(P \circ R_1, P \circ R_2) = d_{\widetilde{\Sigma}}(R_1, R_2)$ for $\forall P \in GL(n)$ and $\forall R_1, R_2 \in \widetilde{\Sigma}$. The simple intuition here is to align the two realizations (bring them as *close* as possible) by sliding one along the fiber it belongs to (see Figure 1). The main difficulty is that (due to non-compactness of $GL(n)$) constructing such a $GL(n)$-invariant distance is difficult and complicated, although not impossible (see [2] and references therein). Our approach is principally the same but first we try to reduce the group $GL(n)$ (in a specific sense) to a compact group as described next.

A. Standardization and Balancing of Realizations

Working with a noncompact group such as $GL(n)$ brings about various theoretical and computational challenges. One might wonder if we could somehow replace $GL(n)$ with its compact subgroup $O(n)$, the subgroup of orthogonal matrices. The answer is positive and in a general setting it is called *reduction of structure group*, a notion which has a precise meaning in differential geometry [21]. In the context of control applications, certain notions of realization *balancing* can be linked to the reduction of structure group [2]. However, the standard diagonal balancing as defined by Moore [23] and others does *not* conform with the reduction of structure group. Nevertheless, a slightly more general form of balancing as introduced in [15] and [27] does. Here, one makes the controllability and observability Gramians *equal* but does *not* enforce diagonality. Define the set of asymptotically stable balanced realizations as

$$\widetilde{\mathcal{O}\Sigma}^{\text{min,a,bl}}_{m,n,p} = \{(A, B, C) \in \widetilde{\Sigma}^{\text{min,a}}_{m,n,p} | W_{\text{o}} = W_{\text{c}} \succ 0\}, \quad (4)$$

and the set of k-balanced realizations ($k \geq n$) as

$$\widetilde{\mathcal{O}\Sigma}^{\text{min,bl},k}_{m,n,p} = \{(A, B, C) \in \widetilde{\Sigma}^{\text{min}}_{m,n,p} | W_{\text{o},k} = W_{\text{c},k} \succ 0\}, \quad (5)$$

where $X \succ 0$ means that the matrix X is positive definite. Notice that if R belongs to $\widetilde{\mathcal{O}\Sigma}^{\text{min,a,bl}}_{m,n,p}$ so does $Q \circ R$ for every $Q \in O(n)$. In fact, the reverse can be shown too, if R and $P \circ R$ belong to $\widetilde{\mathcal{O}\Sigma}^{\text{min,a,bl}}_{m,n,p}$ for some $P \in GL(n)$, then $P \in O(n)$.[3] The same holds for $\widetilde{\mathcal{O}\Sigma}^{\text{min,bl},k}_{m,n,p}$. One can show that $\widetilde{\mathcal{O}\Sigma}^{\text{min,a,bl}}_{m,n,p}$ and $\widetilde{\mathcal{O}\Sigma}^{\text{bl},k}_{m,n,p}$ are smooth submanifolds (subbundles) of $\widetilde{\Sigma}^{\text{min,a}}_{m,n,p}$ and $\widetilde{\Sigma}^{\text{min}}_{m,n,p}$, respectively. More importantly, we have $\Sigma^{\text{min,a}}_{m,n,p} = \widetilde{\mathcal{O}\Sigma}^{\text{min,a,bl}}_{m,n,p} / O(n)$ and $\Sigma^{\text{min}}_{m,n,p} = \widetilde{\mathcal{O}\Sigma}^{\text{min,bl},k}_{m,n,p} / O(n)$, where equality is understood in the sense of diffeomorphism (see [2] for more details). Informally, we could get rid of the non-compact part of $GL(n)$ without causing any topological complications. This latter fact enables us to easily define group action induced distances on the space of LTI systems (the bottom space) using commonly used distances on the space of realizations (the top space). Other forms of balancing and reduction of structure are possible (see [2]). We call all these transformations *standardization* or *orthogonalization* of the realization space or bundle.

1) More on Balancing: Notice that we explicitly distinguished between diagonally balanced realizations (the often used form of balancing) and balanced realizations. A diagonally balanced realization is balanced in this more general sense but not vice versa. diagonal balancing is *not* a *topologically* natural operation and, in general, has no useful topological meaning e.g., the set of diagonally balanced realizations does *not* form a *smooth submanifold* of the total realization bundle $\widetilde{\Sigma}^{\text{min,a}}_{m,n,p}$. In diagonal balancing (or in diagonally balanced canonical forms, see e.g., [25]) one tries to remove the group action ambiguity from realizations by assigning exactly one realization to every system. However, an important topological fact is that *no smooth* canonical form on the manifold $\widetilde{\Sigma}^{\text{min}}_{m,n,p}$ (or $\widetilde{\Sigma}^{\text{min,a}}_{m,n,p}$) exists (when $\min(m, p) > 1$, see [14], [7]). Therefore, the assignment cannot be smooth globally and at certain points it loses continuity (the non-uniqueness of diagonally balanced realizations has been a source of complication since the early days [23]). The unfortunate result of this topological obstruction is that one cannot define a bonafide *distance* between systems by simply comparing their (discontinuous) respective canonical forms. On the other hand, as mentioned before, balanced realizations form smooth submanifolds and they can be used to define group action induced distances, which we call the alignment distances.

B. The Alignment Distance

The alignment distance is a group action induced distance which can be defined in a very general setting [2], [3]. However, here we limit ourselves to some specific choices. The starting point is a unitary-invariant distance on the space of *realizations*. For example, we consider the Frobenius norm based distance:

$$d_F^2(R_1, R_2) = \|A_1 - A_2\|_F^2 + \|B_1 - B_2\|_F^2 + \|C_1 - C_2\|_F^2, \quad (6)$$

where $R_i = (A_i, B_i, C_i), i = 1, 2$. Then given a standardized realization space, the realization alignment problem is simply defined as aligning two standardized realizations by an orthogonal state-space change of basis so as to minimize the above distance. Specifically we define:

Definition 1 (Alignment Distance): Let $(\widetilde{\mathcal{O}\Sigma}_{m,n,p}, \Sigma_{m,n,p})$ denote either $(\widetilde{\mathcal{O}\Sigma}^{\text{min,a,bl}}_{m,n,p}, \Sigma^{\text{min,a}}_{m,n,p})$ or $(\widetilde{\mathcal{O}\Sigma}^{\text{min,bl},k}_{m,n,p}, \Sigma^{\text{min}}_{m,n,p})$. Let M_1 and M_2 be two systems in $\Sigma_{m,n,p}$. The alignment distance subordinate to (standardization) $\widetilde{\mathcal{O}\Sigma}_{m,n,p}$ is defined as

$$d_{\Sigma_{m,n,p}}(M_1, M_2) = \min_{Q \in O(n)} d_F(Q \circ R_1, R_2), \quad (7)$$

where $R_i (i = 1, 2)$ is any realization of M_i in $\widetilde{\mathcal{O}\Sigma}_{m,n,p}$. The above minimization problem is called the realization alignment problem subordinate to $\widetilde{\mathcal{O}\Sigma}_{m,n,p}$.

[3] This is an important property whose proof is not trivial and is essentially equivalent to the notion of reduction of structure group.

It can be shown that the alignment distance is a true distance, i.e., it is symmetric, positive definite and obeys the triangle inequality (see [2]). Computing the alignment distance amounts to a non-convex optimization on the compact manifold of orthogonal matrices. The problem actually is closely related to (however more general than) the well-studied Procrustes analysis appearing in statistics [13]. Using tools from optimization on manifolds efficient algorithms can devised, where the likelihood of finding a global minimizer can be increased by using multistarting strategies or other approaches (see [18], [1]).

Figure 1 depicts the steps in computing the alignment distance: given two systems M_1 and M_2 (with possibly non-standardized realizations) and the standardized subbundle $\widetilde{\mathcal{O}\Sigma}_{m,n,p}$, one first standardizes the realizations (e.g., via a diagonal balancing algorithm in the case of $(\widetilde{\mathcal{O}\Sigma}^{\min,a,bl}_{m,n,p}, \Sigma^{\min,a}_{m,n,p})$) to get standardized realizations R_1 and R_2 in the standardized subbundle $\widetilde{\mathcal{O}\Sigma}_{m,n,p}$, then one aligns the two standardized realizations R_1 and R_2, i.e., solves (7) by sliding one realization, say R_1 with an orthogonal matrix Q to the closest d_F-distance from R_2, and then finds the distance $d_{\Sigma_{m,n,p}}(M_1, M_2) = d_F(Q \circ R_1, R_2)$.

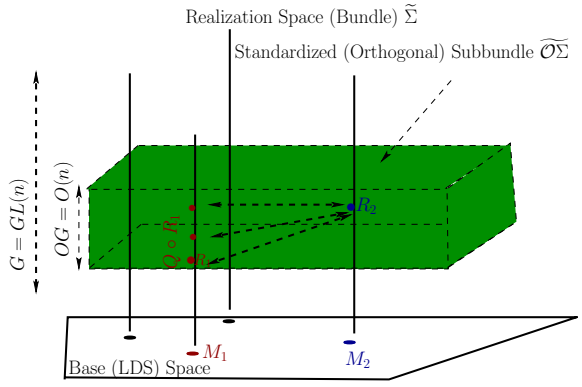

Fig. 1. Computing the alignment distance between to systems M_1 and M_2, see (7).

IV. Model Order Reduction in the Alignment Distance: Problem Formulation

A. Boundary of the Manifold of Balanced Minimal Realizations

In mathematical terms, we want to extend the alignment distance in (7) to the space of systems of order less than or equal to n, i.e., we consider this space as the metric completion of the manifold of systems of (minimal) order n. Due to space limitation we leave a rigorous construction and discussion (including topological aspects) to a later work, and we follow a more applied and intuitive approach here. The next proposition, whose proof is straightforward, shows that the boundary points (taken in the distance d_F) are simply balanced realizations with singular Grammians.

Proposition 2: Let $\overline{\widetilde{\mathcal{O}\Sigma}}^{\min,bl,k}_{m,n,p}$ $(k \geq n)$ denote the closure of $\widetilde{\mathcal{O}\Sigma}^{\min,bl,k}_{m,n,p}$ in $\widetilde{\mathcal{L}}_{m,n,p}$ and $\overline{\widetilde{\mathcal{O}\Sigma}}^{\min,a,bl}_{m,n,p}$ denote the closure of $\widetilde{\mathcal{O}\Sigma}^{\min,a,bl}_{m,n,p}$ in $\mathcal{L}^{a}_{m,n,p}$. Then we have:

$$\overline{\widetilde{\mathcal{O}\Sigma}}^{\min,bl,k}_{m,n,p} = \{R \in \widetilde{\mathcal{L}}_{m,n,p} \mid W_{o,k} = W_{c,k} \succeq 0\}, \quad (8)$$

and

$$\overline{\widetilde{\mathcal{O}\Sigma}}^{\min,a,bl}_{m,n,p} = \{R \in \widetilde{\mathcal{L}}^{a}_{m,n,p} \mid W_o = W_c \succeq 0\}. \quad (9)$$

The next proposition characterizes the realizations on the boundary. The proof of the proposition is omitted due to space limitation.

Proposition 3: (i) On the boundary of $\widetilde{\mathcal{O}\Sigma}^{\min,a,bl}_{m,n,p}$ any realization of minimal order not larger than $r < n$ is of the form $Q \circ \bar{R}$ where $Q \in O(n)$ and $\bar{R} = (\bar{A}, \bar{B}, \bar{C})$ are of the form

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & 0 \\ 0 & \bar{A}_{22} \end{bmatrix}, \bar{B} = \begin{bmatrix} \bar{B}_{11} \\ 0 \end{bmatrix}, \bar{C} = \begin{bmatrix} \bar{C}_{11} & 0 \end{bmatrix}, \quad (10)$$

where the (sub-)realization $\bar{R}_{11} = (\bar{A}_{11}, \bar{B}_{11}, \bar{C}_{11}) \in \overline{\widetilde{\mathcal{O}\Sigma}}^{\min,a,bl}_{m,r,p}$ is balanced and $\bar{A}_{22}$ is asymptotically stable. If the minimal order is r then $\bar{R}_{11} \in \widetilde{\mathcal{O}\Sigma}^{\min,a,bl}_{m,r,p}$; (ii) On the boundary of $\widetilde{\mathcal{O}\Sigma}^{\min,bl,k}_{m,n,p}$ any realization of minimal order not larger than r is of the form (10) except that $\bar{R}_{11} = (\bar{A}_{11}, \bar{B}_{11}, \bar{C}_{11}) \in \overline{\widetilde{\mathcal{O}\Sigma}}^{\min,bl}_{m,r,p}$ and if the minimal order is exactly r then $R_{11} \in \widetilde{\mathcal{O}\Sigma}^{\min,bl,k}_{m,r,p}$.

B. Model Reduction in the Alignment Distance

We define model order reduction in the alignment distance as follows:

Definition 4 (Model Reduction in the Alignment Distance): Let $(\widetilde{\mathcal{O}\Sigma}_{m,n,p}, \Sigma_{m,n,p})$ denote either $(\widetilde{\mathcal{O}\Sigma}^{\min,a,bl}_{m,n,p}, \Sigma^{\min,a}_{m,n,p})$ or $(\widetilde{\mathcal{O}\Sigma}^{\min,bl,k}_{m,n,p}, \Sigma^{\min}_{m,n,p})$. Let M be an LTI system in $\Sigma_{m,n,p}$ with realization $R \in \widetilde{\mathcal{O}\Sigma}_{m,n,p}$. Then the r^{th}-order model order reduction in the alignment distance (subordinate to $\widetilde{\mathcal{O}\Sigma}_{m,n,p}$) is defined as

$$\inf_{Q \in O(n), \bar{R}} d_F(Q \circ R, \bar{R}) \quad (11)$$

where $\bar{R}$ is a realization of minimal order at most r on the boundary of $\widetilde{\mathcal{O}\Sigma}_{m,n,p}$ of the form (10) in Proposition 3 and we call such a realization a feasible realization. If a feasible realization achieving the infimum exists we call this realization the reduced order realization or the solution.

The following proposition (which could possibly be improved) gives a-priori existence of solution(s) to the problem:

Proposition 5 (A-priori existence of solution): (i) In the case of $(\widetilde{\mathcal{O}\Sigma}^{\min,bl,k}_{m,n,p}, \Sigma^{\min}_{m,n,p})$ the problem in (11) always has a solution, i.e., there exists a realization $\bar{R}$ of order at most r on the boundary of $\widetilde{\mathcal{O}\Sigma}^{\min,bl,k}_{m,n,p}$ achieving the infimum; (ii) In the case of $(\widetilde{\mathcal{O}\Sigma}^{\min,a,bl}_{m,n,p}, \Sigma^{\min,a}_{m,n,p})$ a realization achieving the infimum always exists, and such a realization is either asymptotically stable (and solution to the problem) or it is only stable (i.e., it has pole(s) on the unit circle).

Proof: For (i) first note that $\overline{\widetilde{\mathcal{O}\Sigma}}^{\min,bl,k}_{m,n,p}$ is a closed set in $\widetilde{\mathcal{L}}_{m,n,p}$ (hence a complete metric space). Also note

that if $\bar{R} = (\bar{A}, \bar{B}, \bar{C})$ is a feasible realization for which one of $\|\bar{A}\|_F$, $\|\bar{B}\|_F$, or $\|\bar{C}\|_F$ is larger than $2(\|A\|_F + \|B\|_F + \|C\|_F)$, then $d_F(R, \bar{R}) > d_F(R, 0)$. This means that the feasible set for the minimization problem (11) can be considered as a closed and bounded (hence compact) set (note that $O(n)$ is compact). Thus a solution achieving the infimum exists. For (ii) the situation is similar in terms of boundedness of the feasible set; however, in this case the set of feasible realizations is open in $\widetilde{\mathcal{L}}_{m,n,p}$ with boundary points being realizations which are limits of realizations in $\widetilde{\mathcal{OS}}^{\text{min,a,bl}}_{m,n,p}$ and have poles on the unit circle. The statement is a consequence of this fact. ∎

Strictly speaking the model reduction problem above is not expressed explicitly in terms of the alignment distance between systems (primarily for convenience). However, if we define the quotient space $\overline{\Sigma}_{m,n,p} = \widetilde{\overline{\mathcal{OS}}}_{m,n,p}/O(n)$, then the model reduction problem can formulated as

$$\min_{\bar{M}} d_{\bar{\Sigma}_{m,n,p}}(M, \bar{M}), \tag{12}$$

where $\bar{M}$ is a system of order at most r (on the boundary of $\overline{\Sigma}_{m,n,p}$) and $d_{\bar{\Sigma}_{m,n,p}}(\cdot,\cdot)$ is the alignment distance on $\overline{\Sigma}_{m,n,p}$.

V. An Alternating Minimization Algorithm: Align, Truncate, & Project (ATP)

In this section we give an algorithm for solving the model order reduction problem (11) using alternating minimization between Q and $\bar{R} = (\left[\begin{smallmatrix} \bar{A}_{11} & 0 \\ 0 & \bar{A}_{22} \end{smallmatrix}\right], [\begin{smallmatrix} \bar{B}_{11} & 0 \end{smallmatrix}], \left[\begin{smallmatrix} \bar{C}_{11} \\ 0 \end{smallmatrix}\right])$. We limit our derivation to $(\widetilde{\mathcal{OS}}^{\text{min,a,bl}}_{m,n,p}, \Sigma^{\text{min,a}}_{m,n,p})$, and leave the other case to a later work.

Let Q and $\bar{R} \in \widetilde{\mathcal{OS}}^{\text{min,a,bl}}_{m,n,p}$ solve the minimization in (11). If Q is fixed in the minimization $\min_{\bar{R}} d_F(Q \circ R, \bar{R})$, where $\bar{R}$ is in the form (10), then we must have $\bar{A}_{22} = (Q^\top A Q)_{22}$. Note that the top sub-realization of $\bar{R}$, $\bar{R}_{11}$, solves

$$\min_{\bar{R}_{11} \in \widetilde{\overline{\mathcal{OS}}}^{\text{min,a,bl}}_{m,r,p}} d_F((Q \circ R)_{11}, \bar{R}_{11}). \tag{13}$$

In the above we are slightly abusing the notation since here $d_F(\cdot,\cdot)$ is defined on the truncated realizations (of order r instead of n as in (6)). The above problem is nothing but the projection (in the d_F distance) of the truncated part or top sub-realization of $Q \circ R$ (namely, $(Q \circ R)_{11}$) onto the space $\widetilde{\overline{\mathcal{OS}}}^{\text{min,a,bl}}_{m,r,p}$. The scheme of the iterative algorithm is clear now: We start with an initial guess $\bar{R}^0$, solve the *alignment* problem to find Q^1, *truncate* $Q^1 \circ \bar{R}^0$ to get the $(Q^1 \circ \bar{R}^0)_{11}$ and $\bar{A}^1_{22}$ in $\bar{R}^1$, then *project* $(Q^1 \circ \bar{R}^0)_{11}$ onto $\widetilde{\overline{\mathcal{OS}}}^{\text{min,a,bl}}_{m,r,p}$ to get $\bar{R}^1_{11}$, and repeat the three steps to generate $\bar{R}^k_{11}$ ($k \geq 1$) till convergence. We call this algorithm the align, truncate, and project (ATP) algorithm.

The main computational challenge here is the step of projection onto the space of balanced realizations (i.e., solving (13)). An important fact is that, in discrete-time, sub-realizations of a balanced realization (even if diagonally balanced) are not, in general, balanced. It is interesting to recall that in the continuous-time case and for a diagonally balanced realization the sub-realizations are balanced (see [23]). Therefore, in this case, we need to device an algorithm for the projection. Nevertheless, we mention that in certain cases no projection might be needed, e.g., for balanced realization R, if the A matrix is symmetric then we have $A_{11} = A_{11}^\top$, $BB^\top = C^\top C$, which implies $B_{11}B_{11}^\top = C_{11}^\top C_{11}$, which in turn implies that for $R_{11} = (A_{11}, B_{11}, C_{11})$ observability and controllability Grammians are equal (consider 2 with $W_o = W_c$ and A symmetric). As a practical note, also we mention that in some cases the first step of alignment might give a good enough approximate solution; or as an approximation one might try to simply re-balance the truncated realization using standard balancing (change of coordinate) algorithms (but such a re-balancing obviously need not to coincide projection (13).

A. Projection Algorithm

Here, we use the alternating direction method of multiplies (ADMM) and augmented Lagrangian to solve (13) (see [5, Section 4.2] and [6]). We relax the (asymptotic) stability constraint on the sought realization $\bar{R}$, i.e., we solve a constrained optimization in which the only constraint is equality of Gramians induced from the discrete-time Lyapunov equations (2). Note that the Lyapunov equations for Gramians might have solution even if the system is unstable and the actual Gramians do not exist. In our experiments all the found projections from asymptotically stable systems remain asymptotically stable and we conjecture that this relaxation generically yields the exact solution.

We consider the constraint $\tilde{W}_c - \tilde{W}_o = 0$ and the following augmented Lagrangian (notice the re-labeling of variables, cf. (13))

$$F_\rho(\tilde{R}; R, Y) = d_F^2(R, \tilde{R}) + g_\rho(\tilde{R}) \tag{14}$$

where

$$g_\rho(\tilde{R}) = \text{tr}(Y^\top(\tilde{W}_c - \tilde{W}_o)) + \frac{\rho}{2}\|\tilde{W}_c - \tilde{W}_o\|_F^2 \tag{15}$$

where $Y \in \mathbb{R}^{r \times r}$ is the Lagrange multiplier, $\rho > 0$ is the penalty parameter, $\tilde{W}_o(\tilde{R})$ and $\tilde{W}_c$ are the Gramians of the sought balanced realization $\tilde{R} = (\tilde{A}, \tilde{B}, \tilde{C})$. Each itertation of ADMM consists of the following internal minimization steps and the Lagrangian and penalty parameter updates:

1) $\tilde{A}^{l+1} = \text{argmin}_{\tilde{A}} f_1(\tilde{A}; \tilde{B}^l, \tilde{C}^l)$, where $f_1(\tilde{A}; \tilde{B}, \tilde{C}) = \|A - \tilde{A}\|_F^2 + g_{\rho^l}(\tilde{A}, \tilde{B}^l, \tilde{C}^l)$,
2) $\tilde{B}^{l+1} = \text{argmin}_{\tilde{B}} f_2(\tilde{B}; \tilde{A}^{l+1}, \tilde{C}^l)$, where $f_2(\tilde{B}; \tilde{A}, \tilde{C}) = \|B - \tilde{B}\|_F^2 + g_{\rho^l}(\tilde{A}^{l+1}, \tilde{B}, \tilde{C}^l)$
3) $\tilde{C}^{l+1} = \text{argmin}_{\tilde{C}} f_3(\tilde{C}; \tilde{A}^{l+1}, \tilde{B}^{l+1})$, where $f_3(\tilde{C}; \tilde{A}, \tilde{B}) = \|C - \tilde{C}\|_F^2 + g_{\rho^l}(\tilde{A}^{l+1}, \tilde{B}^{l+1}, \tilde{C})$
4) $Y^{(l+1)} = Y^{(l)} + \rho^l(\tilde{W}_c^{(l+1)} - \tilde{W}_o^{(l+1)})$
5) increase ρ^l to ρ^{l+1} (e.g., $\rho^{l+1} = \beta\rho^l$ where $\beta > 1$ is a constant).

Generally (i.e., in a non-convex problem) ρ^l needs to gradually increase to pass an *unknown* threshold (see [5, Section 4.2]). For the internal minimization steps we use simple gradient descent with Armijo's line search (although, generally,

for multiplier methods a higher accuracy method such as the Newton method is prescribed [5]). To perform the gradient descent we need to find the gradients of the functions $\tilde{A} \mapsto f_1(\tilde{A};\tilde{B},\tilde{C})$, $\tilde{B} \mapsto f_2(\tilde{B};\tilde{A},\tilde{C})$, and $\tilde{C} \mapsto f_3(\tilde{C};\tilde{A},\tilde{B})$ defined above. We briefly describe the basic ingredients,and more implementational details will appear in the actual code which is available online.[4] We need to calculate the derviates of the Gramians $\tilde{W}_o$ and $\tilde{W}_c$ with respect to matrices $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$. By using standard vectorization and Kronecker product $\otimes$ we can get the following relations from which the desired gradients can be found easily:

$$\begin{aligned}
\overrightarrow{d\tilde{W}_c} &= (I_{n^2} - \tilde{A}\otimes\tilde{A})^{-1}\big((\tilde{A}\tilde{W}_c \otimes I_n) + (I_n \otimes \tilde{A}\tilde{W}_c)H_{nn}\big)\overrightarrow{d\tilde{A}} \\
\overrightarrow{d\tilde{W}_o} &= (I_{n^2} - \tilde{A}^\top\otimes\tilde{A}^\top)^{-1}\big((I_n\otimes\tilde{A}^\top\tilde{W}_o) + (\tilde{A}^\top\tilde{W}_o\otimes I_n)H_{nn}\big)\overrightarrow{d\tilde{A}} \\
\overrightarrow{d\tilde{W}_c} &= (I_{n^2} - \tilde{A}\otimes\tilde{A})^{-1}\big((\tilde{B}\otimes I_n) + (I_n \otimes \tilde{B})H_{nm}\big)\overrightarrow{d\tilde{B}} \\
\overrightarrow{d\tilde{W}_o} &= (I_{n^2} - \tilde{A}^\top\otimes\tilde{A}^\top)^{-1}\big((I_n \otimes \tilde{C}^\top) + (\tilde{C}^\top \otimes I_n)H_{pn}\big)\overrightarrow{d\tilde{C}}
\end{aligned} \tag{16}$$

Here H_{mn} is the $mn\times mn$ commutation matrix, i.e., it solves $\overrightarrow{X^\top} = H_{mn}\overrightarrow{X}$ for any $m \times n$ matrix X, $\overrightarrow{\tilde{A}}$ denotes the column-wise stacked (vectorized) version of matrix $\tilde{A}$, and $d\tilde{A}$ is the so-called differential of $\tilde{A}$ (a matrix of the same size as $\tilde{A}$). In each of the above equations the matrix that relates the vectorized differential on the right hand-side to the vectorized differential on the left hand-side is the respective derivative, e.g., the derivative $\frac{\overrightarrow{d\tilde{W}_c}}{\overrightarrow{d\tilde{A}}}$ is the $n^2 \times n^2$ matrix $(I_{n^2} - \tilde{A}^\top\otimes\tilde{A}^\top)^{-1}\big((I_n\otimes\tilde{A}^\top\tilde{W}_o) + (\tilde{A}^\top\tilde{W}_o\otimes I_n)H_{nn}\big)$, and so forth. As an example the gradient of $f_1(\tilde{A};\tilde{B},\tilde{C})$ with respect to $\tilde{A}$ can be calculated as:

$$\begin{aligned}
\operatorname{grad}_{\tilde{A}} f_1(\tilde{A};\tilde{B},\tilde{C}) &= 2(\overrightarrow{\tilde{A}}^\top - \overrightarrow{\tilde{A}}^\top) + \overrightarrow{Y}^\top\Big(\frac{\overrightarrow{d\tilde{W}_c}}{\overrightarrow{d\tilde{A}}} - \frac{\overrightarrow{d\tilde{W}_o}}{\overrightarrow{d\tilde{A}}}\Big) + \\
&\rho\Big(\overrightarrow{\tilde{W}_c}\Big(\frac{\overrightarrow{d\tilde{W}_c}}{\overrightarrow{d\tilde{A}}} - \frac{\overrightarrow{d\tilde{W}_o}}{\overrightarrow{d\tilde{A}}}\Big) - \overrightarrow{\tilde{W}_o}\Big(\frac{\overrightarrow{d\tilde{W}_c}}{\overrightarrow{d\tilde{A}}} - \frac{\overrightarrow{d\tilde{W}_o}}{\overrightarrow{d\tilde{A}}}\Big)\Big).
\end{aligned}$$

Hence, $\tilde{A}$ can be updated as $\overrightarrow{\tilde{A}}^\top \leftarrow \overrightarrow{\tilde{A}}^\top - \mu\operatorname{grad}_{\tilde{A}} f_1(\tilde{A};\tilde{B},\tilde{C})$ where the step-size μ can be choosen according to Armijo's rule (or any other line search method) to ensure decrease at each update. Similar updates for $\tilde{B}$ and $\tilde{C}$ can be derived. With this all the ingredients needed to implement the steps of the ADMM algorithm described above are available.

VI. A-priori Bounds and comparison with balanced truncation

Certain a-priori bounds on model reduction error (measured in L^∞ norm of the error) for methods such as balanced truncation and Hankel norm approximation have appeared in the literature and proven useful (see e.g., [9], [12], [16]). These errors bounds have a simple form in terms of the sum of the $n-r$ smallest singular values of the system. However, it should be noted that there is no sense of optimality in some of these bounds, i.e., in the case of the Hankel norm approximation the bounded error is measured in a sense (L^∞) which is different from the original optimization sense and in the case of balanced truncation, in fact, there is no optimization sense. Here, we derive an a-priori bound on the alignment distance between reduced order system and the original system for model reduction in the alignment distance.

Proposition 6: Let $\lambda_1 \geq \cdots \geq \lambda_r \geq \cdots \geq \lambda_n > 0$ be the singular values of $M \in \Sigma^{\min,a}_{m,n,p}$. Let $\bar{M}$ be a best r^{th} order system $\bar{M}$ approximation of M in the alignment distance $d_{\overline{\Sigma}^{\min,a}_{m,n,p}}$, then we have

$$d_{\overline{\Sigma}^{\min,a}_{m,n,p}}(\bar{M},M) \leq \sqrt{2\Big(\sum_{i=r+1}^{n}\lambda_i\Big)\Big(1+\frac{1}{\lambda_r}\Big)} + \text{P\_Error}, \tag{17}$$

where $d_{\overline{\Sigma}^{\min,a}_{m,n,p}}$ is the alignment distance subordinate to $\widetilde{\mathcal{O}\Sigma}^{\min,a,bl}_{m,n,p}$ and d_F, defined in (7) and P_Error is error in projecting an r^{th} order diagonally balanced truncated realization of M onto $\widetilde{\widetilde{\mathcal{O}\Sigma}}^{\min,a,bl}_{m,r,p}$.

Proof: The main idea of the proof is to bound the error when evaluated for the diagonally balanced truncation of M. Let R and $\bar{R}_T$ be, respectively, a diagonally balanced realization of M and the r^{th} order truncation of R in the form (10) but where the top sub-realization in $\bar{R}_T$ is not balanced. Then let $\bar{R}_{T,\text{bl}}$ be the projected version of $\bar{R}_T$ i.e., where the top sub-realization of $\bar{R}_{T,\text{bl}}$ is the d_F-projection of the top sub-realization of $\bar{R}_T$ onto $\widetilde{\widetilde{\mathcal{O}\Sigma}}^{\min,a,bl}_{m,r,p}$ (i.e., the solution to (13). Then, using the triangle inequality, we have

$$\begin{aligned}
d_{\overline{\Sigma}^{\min,a}_{m,n,p}}\big(\bar{M},M\big) \leq d_F(R,\bar{R}_{T,\text{bl}}) \leq & d_F(R,\bar{R}_T) \\
& + d_F(\bar{R}_T,\bar{R}_{T,\text{bl}})
\end{aligned} \tag{18}$$

Note that $d_F(\bar{R}_T,\bar{R}_{T,\text{bl}})$ is the second term on the right hand side of (17). Next we bound $d_F(R,\bar{R}_T)$. Note that $d_F^2(R,\bar{R}_T) \leq \|A_{12}\|_F^2 + \|A_{21}\|_F^2 + \|B_{21}\|_F^2 + \|C_{12}\|_F^2$. Let $\Lambda = \left[\begin{smallmatrix}\Lambda_1 & 0\\ 0 & \Lambda_2\end{smallmatrix}\right]$ be the diagonally balanced Gramian of M (singular values put in decreasing order). Obviously (from the Lyapunov equations (2)) $B_{21}B_{21}^\top \preceq \Lambda_2$, $C_{12}C_{12}^\top \preceq \Lambda_2$, $A_{21}\Lambda_1 A_{21}^\top \preceq \Lambda_2$, and $A_{12}^\top\Lambda_1 A_{12} \preceq \Lambda_2$. From the first two we have $\|B_{21}\|_F^2 \leq \sum_{i=r+1}^n \lambda_i$ and $\|C_{12}\|_F^2 \leq \|\Lambda_2\|_F$. From the last two we have $\lambda_r A_{21}A_{21}^\top \preceq \Lambda_2$ and $\lambda_r A_{12}^\top A_{12}^\top \preceq \Lambda_2$, and hence $\|A_{21}\|_F^2 \leq \frac{1}{\lambda_r}\sum_{i=r+1}^n\lambda_i$ and $\|A_{12}\|_F^2 \leq \frac{1}{\lambda_r}\sum_{i=r+1}^n\lambda_i$. The result then follows from adding these bounds and taking square root. ∎

This result is not quite satisfactory since the bound on the projection error is not explicitly given. However, note that, as mentioned above, in certain cases the projection error can be assumed to be small or even zero (e.g., in when A is symmetric). Nevertheless, we conjecture that this result can be improved and the projection error term also could be bounded in terms of the Hankel singular values, but we leave further developments to a later work.

A. Connections with Diagonally Balanced Truncation

The first term in the bound (17) (modulo the square root) resembles the well-known bound in the L^∞ norm

[4]Code for this algorithm is available at `http://vision.jhu.edu/code/`. Code for calculating the alignment distance based on the Jacobi method [18] is also available from the same website.

for diagonally balanced truncation methods (cf. [16] for the result in the discrete-time case). In fact, in deriving the bound we used diagonally balanced truncation as a suboptimal solution to bound the error.

As the above discussion suggests, in certain cases Moore's diagonally balanced truncation can be considered as an approximate solution to model reduction in the alignment distance, and in fact our algorithm may be initialized with such a solution. To understand the situation better consider the problem (11) and for the sake of argument assume that R is a diagonally balanced realization of minimal order r, i.e., a realization of the form (10). In this case both R and the sought realization $\bar{R}$ are in the same form and a global solution to (11) is clearly $\bar{R} = R$ and $Q = I$. The key point is that in this case the norms of B_{21} and C_{12} in the bottom or weak sub-realization of R are zero and the optimization (11) which involves three terms in d_F^2 decouples and the global error of zero can be achieved. Therefore, one expects that as long as the mentioned norms are close to zero truncation of a diagonally balanced realization will be a good approximation of model reduction in the alignment distance. The interesting point is that a similar phenomenon in the case of continuous-time systems has been reported in the literature. Specifically, in the continuous-time case the norms of the rows of B and those of the columns of C are equal and they are called *balanced gains* [20]. As a shortcoming of diagonally balanced truncation, it has been argued that diagonally balanced truncation is *blind* to these gains and may result in poor L^2 norm errors [20]. The issue is that the so-called weak sub-realization may have large balanced gains despite having small singular values, and such a sub-realization can influence the behavior of the overall system significantly (a fact which renders the labels 'weak" and 'strong" somewhat bad aliases).

1) The Role of Alignment (Change of Basis): The next example shows that diagonally balanced truncation can result in significantly worse errors than what actual minimization of the alignment distance gives.

Example 7: Consider the diagonally balanced realization $R = (A, B, C)$:

$$A = \left[\begin{smallmatrix} 0.9999 & -0.0010 \\ -0.0010 & 0.9487 \end{smallmatrix}\right], B = \left[\begin{smallmatrix} 0.1026 \\ 0.9997 \end{smallmatrix}\right], C = \left[\begin{smallmatrix} 0.1026 & 0.9997 \end{smallmatrix}\right]. \tag{19}$$

where

$$W_o = W_c = \Lambda = \left[\begin{smallmatrix} 88.7345 & 0 \\ 0 & 9.9931 \end{smallmatrix}\right] \tag{20}$$

Note that in this case since A is symmetric alignment distance reduction does not need a projection step. If we use Moore's truncation of the diagonally balanced realization as an approximation we get the reduced order system with realization $\bar{R}_T = (\left[\begin{smallmatrix} 0.9999 & 0 \\ 0 & 0.9487 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0.1026 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0.1026 & 0 \end{smallmatrix}\right))$ ($(\bar{R}_T)_{11}$ being the minimal first order solution) and error of $d_{\text{bl}} = 1.2783$, whereas with the alignment distance based reduction we get the first order system $\bar{R}_{ATP} = (\left[\begin{smallmatrix} 0.9490 & 0 \\ 0 & 0.9996 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1.0050 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1.0050 & 0 \end{smallmatrix}\right])$ and error $d_{\text{ATP}} = 0.0059$, which is significantly lower than d_{bl}.

In this example, the Hankel singular value of the strong sub-realization is much larger than that of the weak sub-realization, but since the so-called balanced gains of the weak sub-realization are larger than those of the strong sub-realization, simple truncation does not give a good approximation. Here, in analogy, we call the norms of B_{21} and C_{12} as balanced gains. In the example, we indicated the order reduction errors in the alignment distance. If we calculate the L^2 error norm we have $d_{\text{bl},L^2} = 3.1576$ and $d_{\text{ATP},L^2} = 0.5631$, which indicates that the alignment distance model reduction gives better error in this example. However, since our original criterion is different from L^2 this situation does not hold in general. Recall that although a zero (or small) alignment distance between two systems implies zero (or small) distance between their impulse responses, the alignment distance itself is not computed based on direct comparison of the impulse responses. Nevertheless, finding relation between input-output based distances and the alignment distance is a relevant question.

This example shows that in the diagonally balanced form (ordered according to the Hankel singular values) still with a new change of coordinates one can reduce the model reduction error significantly. Informally speaking, although the starting point for the alignment distance model reduction method could be a diagonally balanced realization, the step of *alignment* (or finding the best state-space change basis) in the method does *not* ignore the bottom (or weak) sub-realization and if it has high balanced gains it could influence the optimal solution. It should be noted that in the Hankel norm based model reduction method a similar situation happens [12], where the method starts with a diagonally balanced realization and the weak sub-realization influences or enters the optimal solution.

VII. Simulations

In this section we apply the ATP algorithm to a MIMO system of order $n = 5$ and output-input dimension $(p, m) = (2, 2)$ to obtain a system of order $r = 2$. Consider a system M with a diagonally balanced realization $R = (A, B, C)$:

$$A = \begin{bmatrix} -0.6326 & -0.3671 & 0.0198 & -0.1399 & -0.0390 \\ 0.4070 & 0.4639 & 0.3820 & -0.1468 & -0.0631 \\ 0.1255 & 0.0304 & 0.1101 & -0.1152 & 0.1131 \\ 0.1270 & -0.1599 & -0.1814 & 0.4217 & -0.2466 \\ -0.0228 & 0.0167 & 0.0841 & 0.3995 & -0.2895 \end{bmatrix}, B = \begin{bmatrix} 0.7301 & 0.9050 \\ 0.8392 & 0.4398 \\ -0.6782 & 0.8016 \\ 0.0787 & 0.0659 \\ -0.0172 & -0.0078 \end{bmatrix}, \tag{21}$$

and

$$C = \begin{bmatrix} 0.3269 & 0.3338 & -0.9272 & -0.1553 & -0.0299 \\ 1.0809 & -1.0186 & 0.0223 & -0.0596 & -0.0170 \end{bmatrix}. \tag{22}$$

The singular values of the system are $(\lambda_1 = 2.6951, \lambda_2 = 1.9341, \lambda_3 = 1.1637, \lambda_4 = 0.1755, \lambda_5 = 0.0420)$. We run the ATP algorithm with initial solution as the diagonally balanced truncated realization. In the implementation we use the algorithm in [18] to compute the alignment distance (the alignment step). For the ADMM projection algorithm we use initial value $\rho = 10$ and $\beta = 1.1$. A simple Armjio step-size selection ensures that at each iteration of the internal gradient descents (as described in §V) the cost is reduced and we used a fixed number ($I = 20$) of iterations for gradient descent in each subproblem of the ADMM projection algorithm. We run a total $K = 20$ iterations of ATP. Figure 2 shows the squared of the error in reduction in terms of K. The first point in the graph is the alignment distance error in simply using diagonally balanced truncation. As it can be seen in this

case, the reduction in the error beyond the initial diagonally balanced truncation is not significant (although still tangible). The output of the ATP algorithm i.e., the final reduced order ($r = 2$) (balanced) realization is:

$$\bar{A}_{11} = \begin{bmatrix} -0.6475 & -0.1394 \\ 0.4504 & 0.0980 \end{bmatrix}, \bar{B}_{11} = \begin{bmatrix} 0.9130 & 0.8480 \\ 0.9629 & -0.3501 \end{bmatrix}, \bar{C}_{11} = \begin{bmatrix} 0.4855 & 0.7690 \\ 0.9372 & -0.9619 \end{bmatrix} \tag{23}$$

It is interesting to note that the reduced order system is asymptotically stable like the original system. The singular values of the reduced order system are $\bar{\lambda}_1 = 2.7233$ and $\bar{\lambda}_2 = 1.5454$.

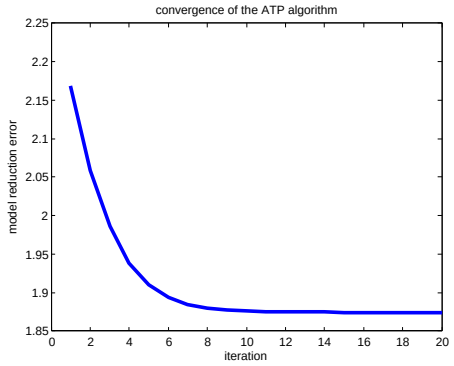

Fig. 2. The performance of the ATP algorithm in an example.

VIII. Conclusions

In this paper we formulated the problem of model reduction in terms of the alignment distance for discrete-time LTI systems. In fact, the problem formulation is quite natural in this setting. We also indicated close a relation between our approach and the popular diagonally balanced truncation method. We derived an algorithm (called ATP) for solving the problem. Several theoretical and computational improvements are possible, which are part of our ongoing research. In the future, we also plan to extend this approach to other classes of systems and perform a comparative study with other methods.

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