# Estimation of Non-Negative ODFs using the Eigenvalue Distribution of Spherical Functions

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## Motivation

- Diffusion magnetic resonance imaging (dMRI) measures the diffusion of water molecules along anatomical fibers in the brain.
- High Angular Resolution Diffusion Imaging (HARDI) measures diffusion by sampling along multiple gradient directions on the sphere ($\mathbb{S}^2$).
- HARDI signals can be represented with a spherical probability distribution function called Orientation Distribution Function (ODF).
- However, due to noise and approximation errors, the estimated ODFs may contain negative values, violating the axioms of a pdf.

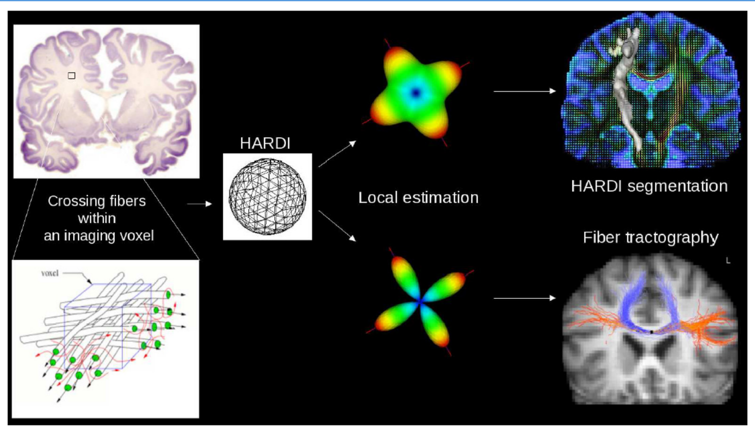



Figure 1. ODF estimation of HARDI signal [Descoteaux 08].

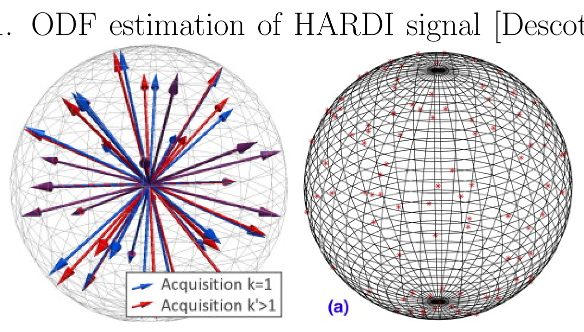



Figure 2. Left: Gradient directions of HARDI signal [Scherrer 11]. Right: Discrete sampling of spherical function [Zhan 10].

## Prior Work

- The method of [1] imposes non-negativity at finitely many directions on a discrete grid on $\mathbb{S}^2$.
- However, this does not guarantee that the estimated ODF be non-negative everywhere on $\mathbb{S}^2$.

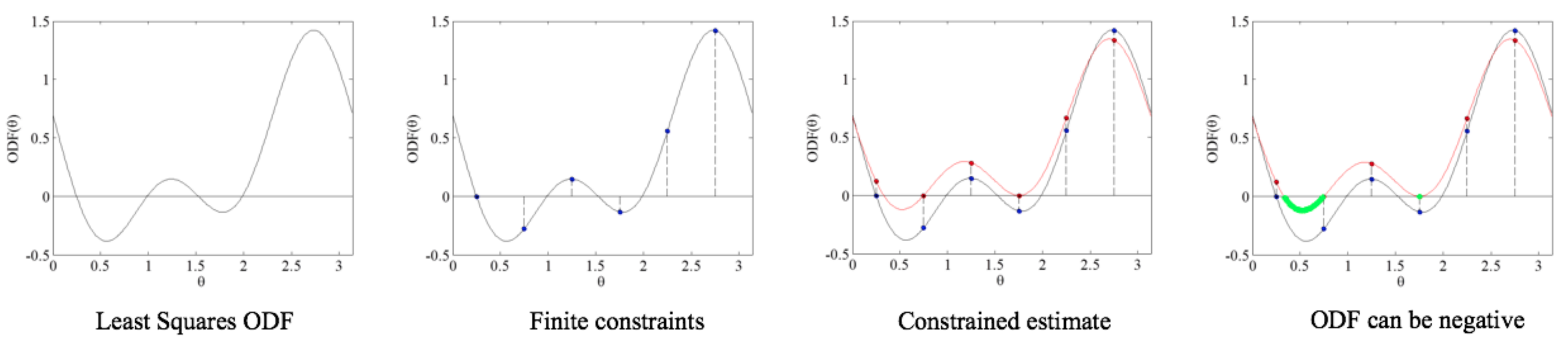



Figure 3. Process of discrete ODF non-negativity optimization.

## Contributions

- We propose a method that enforces non-negativity of ODFs at infinitely many directions on $\mathbb{S}^2$.
- Our approach uses a spherical harmonic (SH) representation of the ODF and enforces non-negativity by applying constraints to the SH coefficients with an iterative semi-definite programming (SDP) algorithm.

## Continuous ODF Estimation Problem Formulation

- Let $s$ = HARDI signal, $\mathbf{B}$ = SH basis matrix $[Y_j]$ and ODF = $f_L = \sum_{j=1}^{R} c_j Y_j$, with $R = \frac{(L+1)(L+2)}{2}$,

Discrete ODF Estimation Problem: $\min_c \|\mathbf{B}c - s\|_2^2$ s.t. $f_L \geq 0$ for finitely many points on $\mathbb{S}^2$ (1)

Continuous ODF Estimation Problem: $\min_c \|\mathbf{B}c - s\|_2^2$ s.t. $f_L \geq 0$ for all points on $\mathbb{S}^2$ (2)

- By imposing constraints on a matrix $T_\ell(f_L)$ [2] formed by linear combinations of zero-padded vector of SH coefficients of $f_L$, we can equivalently constrain the ODF to be non-negative everywhere on $\mathbb{S}^2$ as follows:

$$\min_c \|\mathbf{B}c - s\|_2^2 \quad \text{s.t.} \quad T_\ell(f_L) \succeq 0, \quad \forall \ell \geq L, \tag{3}$$

- But enforcing (3) for infinitely many $\ell$ still requires infinitely many constraints. So instead we look at the behavior of the eigenvalues of $T_\ell(f)$ for all $\ell$.

## Eigenvalue Distribution Theorem on $\mathbb{S}^2$

- **Theorem 1:** Let $\lambda_1^\ell$ be the smallest eigenvalue of $T_\ell(f_L)$. Then:

$$\min f_L = \lim_{\ell \to \infty} \lambda_1^\ell \tag{4}$$

- **Theorem 2:** The sequence of minimum eigenvalues is decreasing, i.e., $\lambda_1^{\ell+1} \leq \lambda_1^\ell$ for all $\ell \geq L$.
- Therefore, enforcing that $f_L \geq 0$, is equivalent to enforcing that $\lambda_1^\ell \geq 0$ for all $\ell \geq L$. We can use the first few values of $\lambda_1^\ell$ to predict the minimum of $f_L$. This leads to an iterative algorithm that alternates between predicting the minimum and imposing a positive-semidefinite constraint.

## Iterative SDP Algorithm

1. Initiate by solving least squares (LS) $\min_c \|\mathbf{B}c - s\|_2^2$ to get coefficents of $f_L$.
2. Calculate $T_L(f_L)$ and extract minimum eigenvalues for $L \leq \ell \leq 20$.
3. Fit curve to eigenvalues to predict $\gamma \triangleq f_L$.
4. Enforce $T_L(f_L)$ - $\gamma\,\mathbf{I} \succeq 0$.
5. Repeat steps 2–4 until $\gamma \geq 0$.

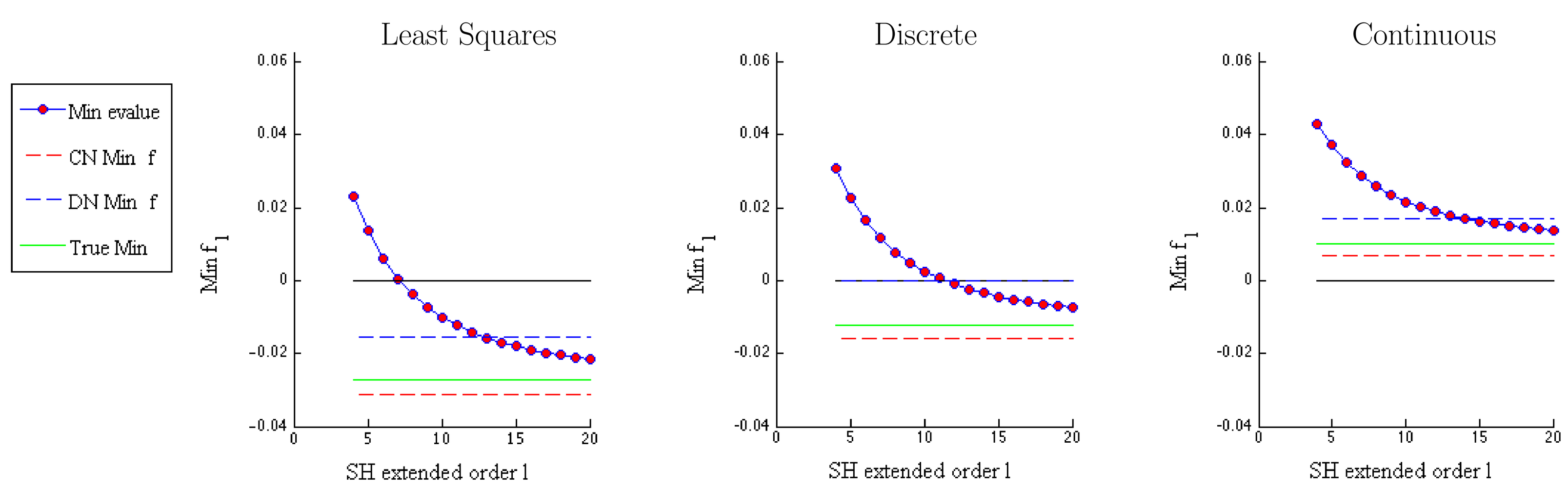
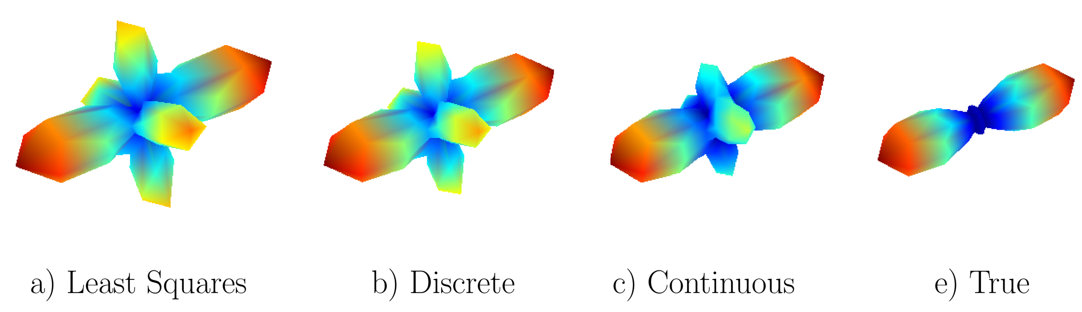



Figure 4. Min eigenvalue of SH order up to $\ell$=20 with true and predicted minima from curve fitting.

## Experiments on Synthetic ODF Fields



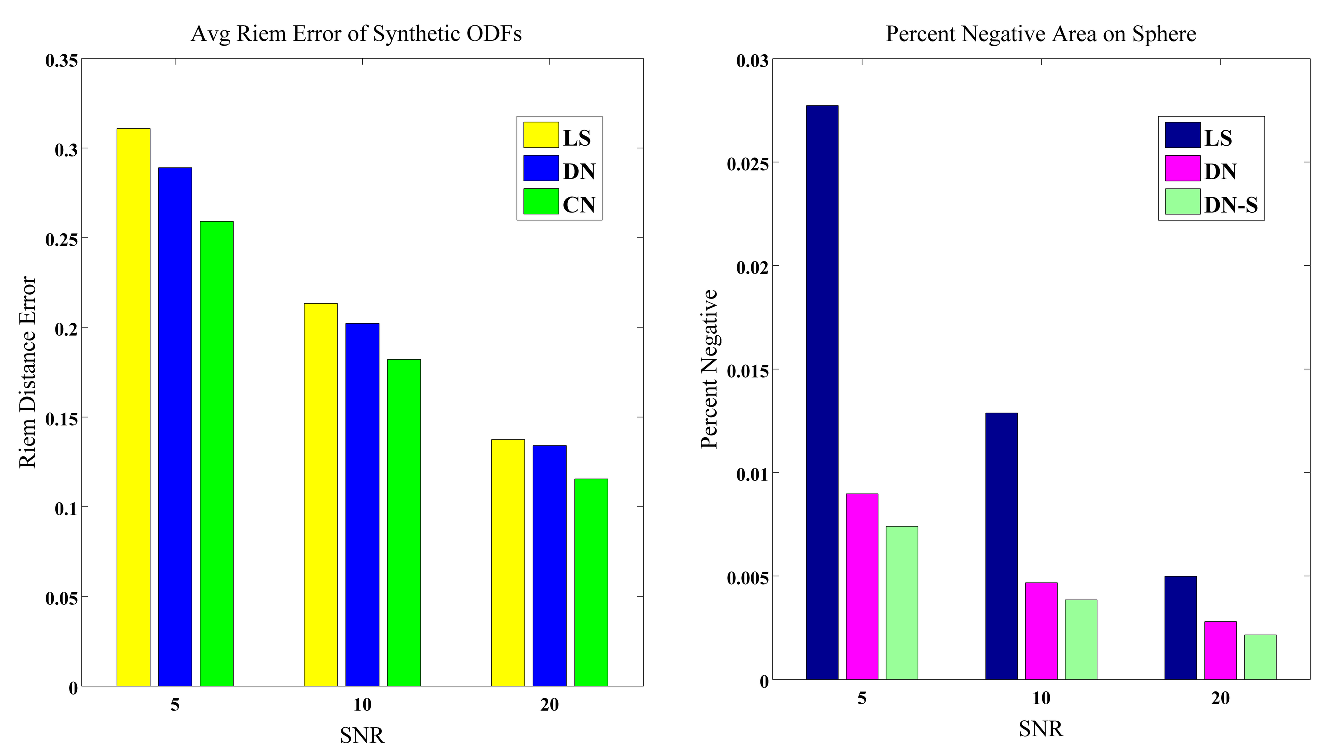




Figure 5. Top: Qualitative ODF comparison with ground truth. Bottom: Quantitative comparison over 100 ODF field.

## Results on Real HARDI Brain Dataset

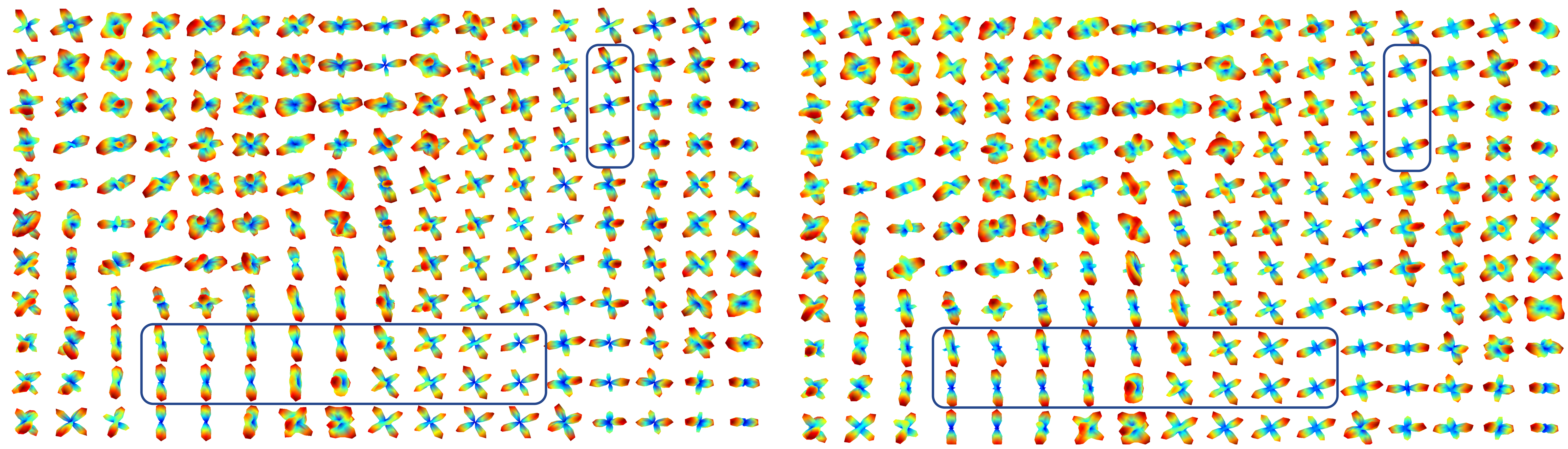


Figure 6. Real HARDI brain ODF field. Left: DN, Right: CN.

## References

[1] Goh, A., Lenglet, C., Thompson, P., Vidal, R.: Estimating Orientation Distribution Functions with Probability Density Constraints and Spatial Regularity. In MICCAI, 2009.

[2] Shirdhonkar, S., Jacobs, D.W.: Non-negative lighting and Specular Object Recognition. In: ICCV, vol. 2, pp. 1323-1330, 2005.

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