NONNEGATIVE ODF ESTIMATION VIA OPTIMAL CONSTRAINT SELECTION

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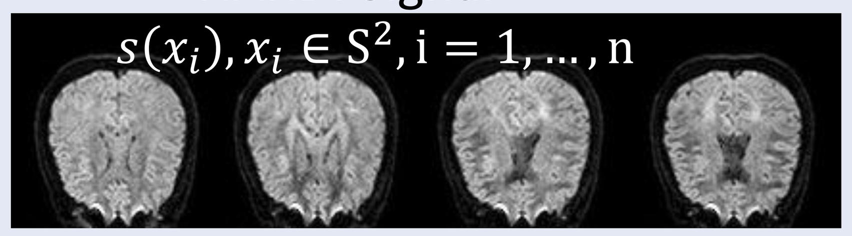
HARDI, QBI and ODF Reconstruction

- *High Angular Resolution Diffusion Imaging* (HARDI) can be used to study the neuronal fiber architecture of the brain in order to study causes of neurological diseases.
- Current *Q-Ball Imaging* (QBI) methods [1,2,3] estimate the *Orientation Distribution Function* (ODF) $p(x),\ x \in S^2$, using a Spherical Harmonics (SH) basis representation for the HARDI signal $s(x), x \in S^2$.

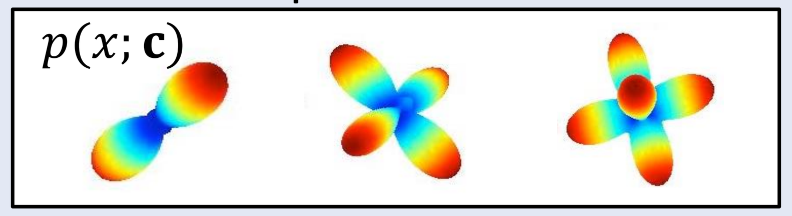
HARDI Signal



$$s(x_i) = \sum_{j=1}^{\infty} c_j Y_j(x_i) \approx \mathbf{Bc}$$

$$\mathbf{c}_{LS} = \text{argmin}_{c \in R^L} \| \mathbf{Bc} - \mathbf{s} \|_2$$

ODFs per voxel



$$p(x;\mathbf{c}) = \frac{1}{4\pi} + \frac{1}{16\pi^2} \sum_{j=1}^{L} c_j \lambda_j l_j Y_j(x)$$

- However, due to signal noise, ODF estimates may contain negative values, which are physically impossible and thus incorrect.

Paper Contributions

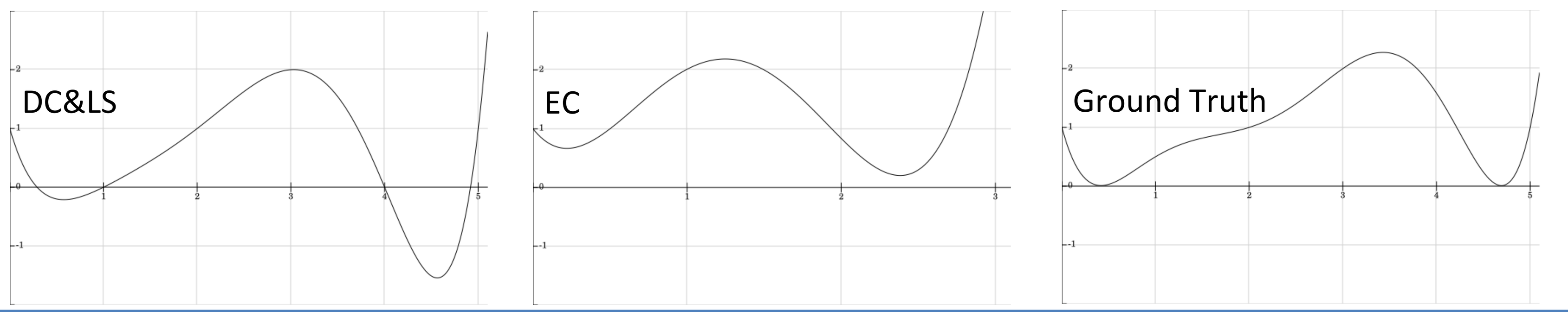
- We propose two methods for estimating nonnegative ODFs from HARDI data.

$$\mathbf{c}_\infty = \text{argmin}_{c \in R^L} \| \mathbf{Bc} - \mathbf{s} \|_2 \quad \text{s.t. } p(x, \mathbf{c}) \geq 0 \ \forall x \in S^2 \qquad (\mathrm{P}_\infty)$$

- The fundamental challenge is that there are infinitely many constraints.

State of the Art

- The discretely constrained (DC) method [4] enforces nonnegativity on a finite grid, which does not guarantee nonnegativity everywhere.
- The eigenvalue constrained (EC) method [5] guarantees nonnegativity, but does not solve P_∞, which can result in a distorted estimate of the ODF.



[1] Tuch, D. (2004) *Q-ball imaging*. Magnetic Resonance in Medicine 52(6), pp. 1358-72.

[2] Descoteaux, M. et al. (2007) *Regularized, fast and robust analytical Q-ball imaging*. Magnetic Resonance in Medicine 58(3), pp. 497–510.

[3] Aganj, I. et al. (2010). *Reconstruction of the orientation distribution function in single- and multiple-shell q-ball imaging within constant solid angle*. Magnetic Resonance in Medicine 64(2), pp. 554–566.

[4] Goh, A. et al. (2009). *Estimating orientation distribution functions with probability density constraints and spatial regularity*. MICCAI, 12(1), pp. 877–885.

[5] Schwab, E. et al. (2012) Estimation of non-negative ODFs using the eigenvalue distribution of spherical functions, MICCAI, 15(2), pp. 322-330.

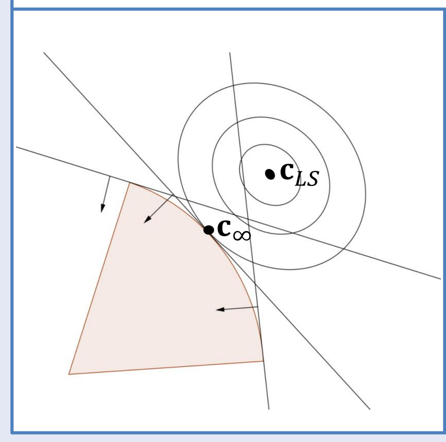
Optimal Constraint Selection

- The key idea behind our approach is that P_∞ is equivalent to

$$\min_{c \in R^L} \| \mathbf{Bc} - \mathbf{s} \|_2 \ \text{ s.t. } \min_{x \in S^2} p(x;\mathbf{c}) \geq \mathbf{0}.$$

- Therefore, one may hope that only one constraint needs to be enforced.
- Indeed, when $p(\cdot\,;\mathbf{c}_\infty)$ has only one zero, x_0, the constraint $p(x_0;\mathbf{c}) \geq \mathbf{0}$ suffices to recover $\mathbf{c}_\infty$.
- To find x_0, we solve an optimization problem on the sphere S^2.

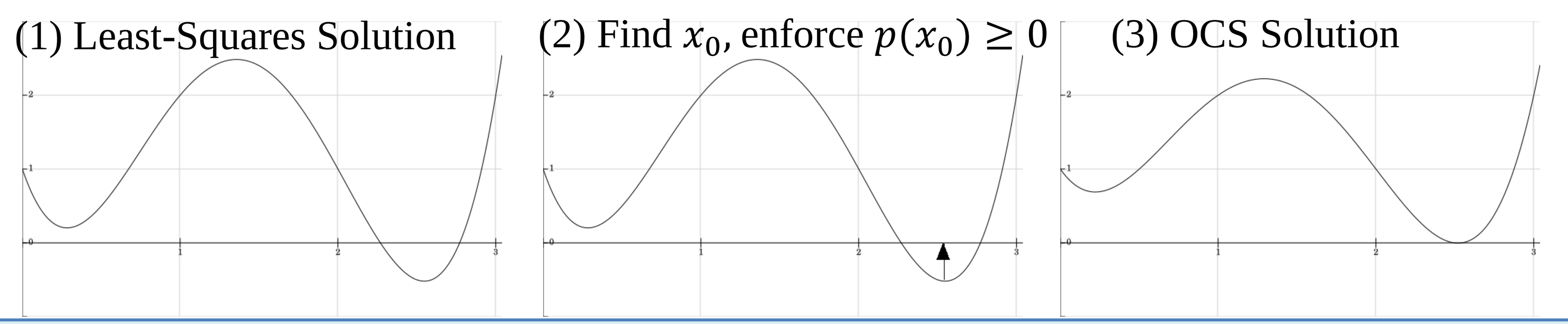
$$x_0 = \text{argmax}_{x \in S^2} \frac{-4\pi - Y(x)^{\mathrm{T}} \mathbf{c}_{LS}}{\sqrt{Y(x)^{\mathrm{T}} (\mathbf{B}^{\mathrm{T}}\mathbf{B})^{-1} Y(x)}} \qquad (\mathrm{OCS}_1)$$



Each line represents one nonnegativity constraint $p(x;\mathbf{c}) \geq \mathbf{0}$. The intersected half-spaces characterize the (nonnegative) ODFs (red). If the optimal ODF $p(\cdot\,;\mathbf{c}_\infty)$ has only one zero, such that there is only one tangent through $\mathbf{c}_\infty$, then this tangent can be found by maximizing the $(\mathbf{B}^{\mathrm{T}}\mathbf{B})^{-1}$-weighted distance (represented by its level sets) to $\mathbf{c}_{LS}$.

- Minimization with only one constraint, $p(x_0;\mathbf{c}) \geq \mathbf{0}$, is done at negligible computational cost in closed form.

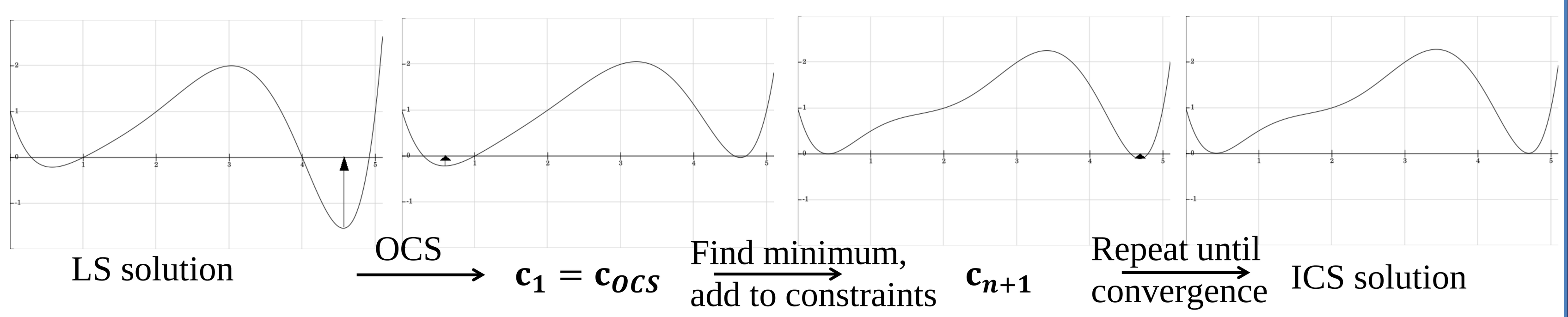
$$\mathbf{c}_{OCS} = \mathbf{c}_{LS} - \frac{4\pi + Y(x_0)^{\mathrm{T}} \mathbf{c}_{LS}}{Y(x_0)^{\mathrm{T}} (\mathbf{B}^{\mathrm{T}}\mathbf{B})^{-1} Y(x_0)} (\mathbf{B}^{\mathrm{T}}\mathbf{B})^{-1} Y(x_0) \qquad (\mathrm{OCS}_2)$$



Note that the minimum in this 1D example is at x = 4.5. However, to obtain nonnegativity everywhere, we enforce $p(4.67) \geq 0$.

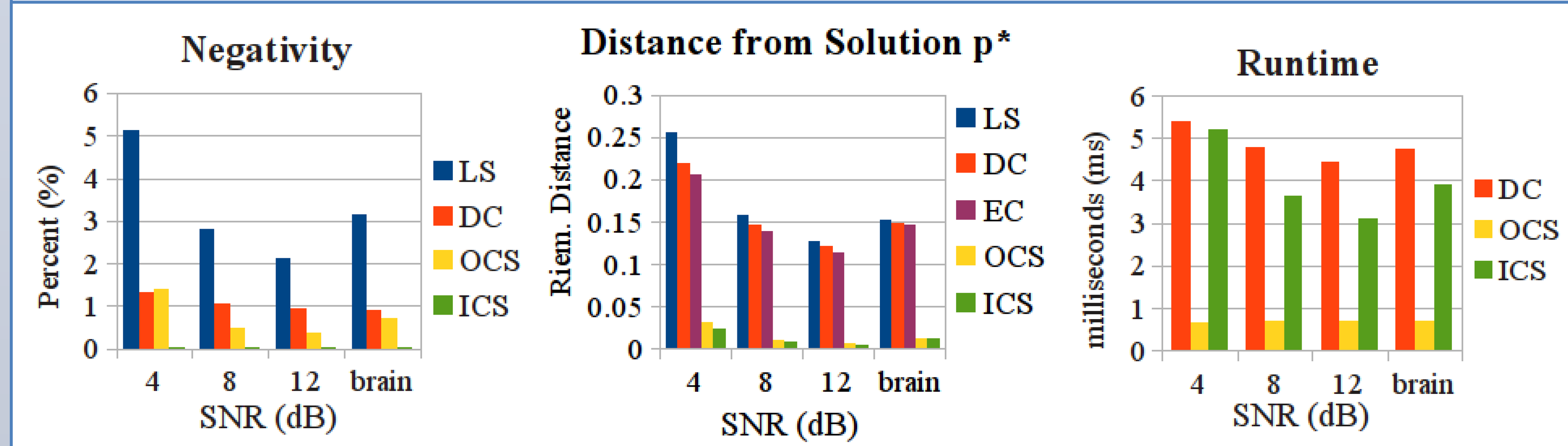
Iterative Constraint Selection

- OCS is only guaranteed to work when $p(\cdot\,;\mathbf{c}_\infty)$ has only one zero; however, with low SNR this assumption may fail.
- In this case, we find a new solution for $\mathbf{c}$ subject to $p(x_1;\mathbf{c}) \geq \mathbf{0}$, where x_1 is the minimum of $p(\cdot\,;\mathbf{c}_{OCS})$.
- Iteratively adding the minimum of the current estimate to the set of constraints generates a converging sequence $\mathbf{c}_n \to \mathbf{c}_\infty$.

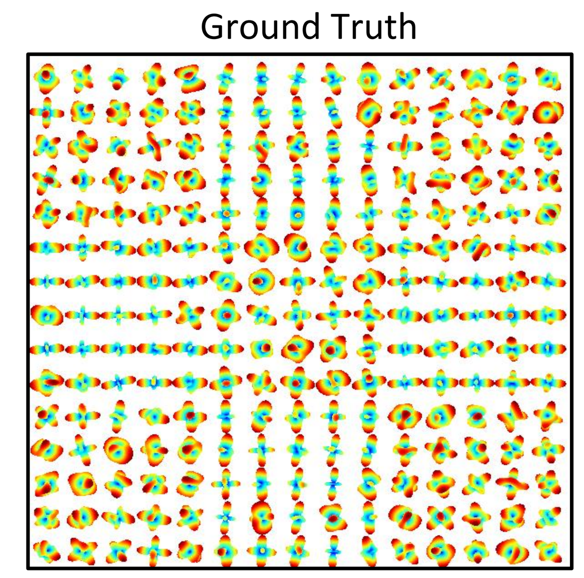


Experiments

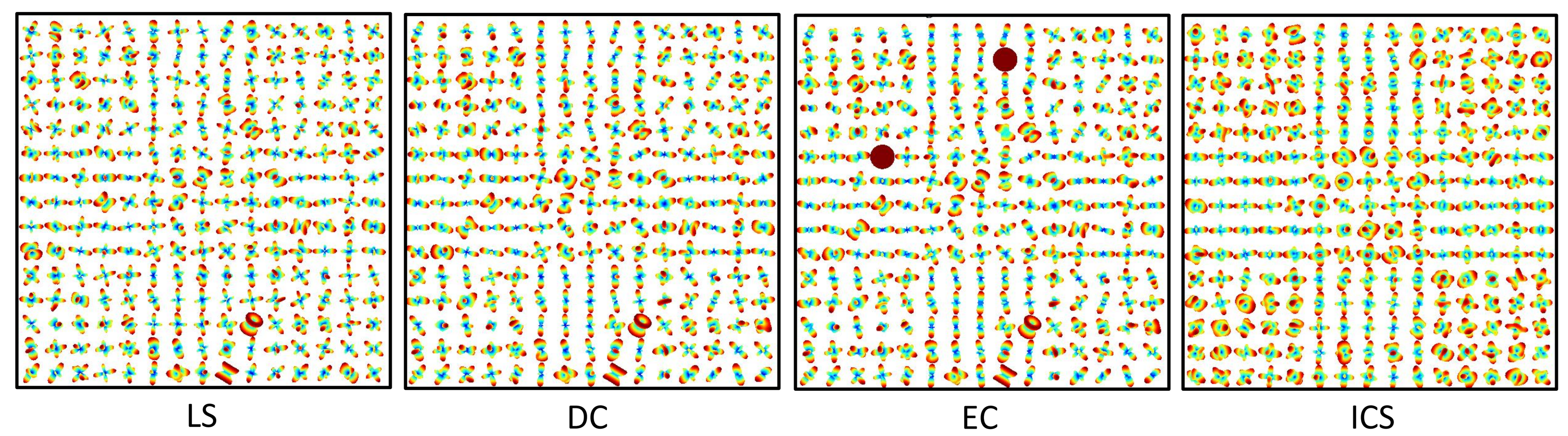
- We compare our methods OCS and ICS with LS, DC, and EC.
- We reconstruct a set of 375 synthetic 1-, 2-, and 3-fiber ODFs distorted with SNR of 4, 8, and 12 dB, as well as a real HARDI brain data set.
- We evaluate the overall performance of each method using three metrics:



- EC always provides nonnegative ODFs, but has a runtime on the order of 10s per ODF and does not provide solutions of P_∞.
- OCS is the fastest method, with a runtime of 0.5ms per ODF.
- Runtime of ICS is comparable to that of DC.
- Overall, ICS outperforms the state-of-the-art EC by producing ODFs that are closer to the optimal nonnegative ODF and improving runtime.



Synthetic fiber field reconstructed with state-of-the-art methods in comparison to the proposed method ICS and the ground truth ODFs.



Conclusions

- We propose two methods for estimating a nonnegative ODF from HARDI data:
  - OCS solves a quadratic problem subject to one constraint and is guaranteed to produce a nonnegative ODF under some condition.
  - ICS iteratively solves a quadratic problem subject to multiple linear constraints and is guaranteed to converge to the correct solution.
- Experiments showed that our methods produce more accurate solutions than prior work at a reduced runtime .

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