Is an Affine Constraint Needed for Affine Subspace Clustering?

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Introduction

- Vision datasets often contain multiple classes, each lying in a low-dimensional subspace.
- In many cases, the subspaces do not pass through the origin, i.e., they are affine.
- Affine subspace clustering (ASC): discover affine subspaces in an unsupervised manner.
- The collection of two lines (i.e., 1D affine subspaces) in \( \mathbb{R}^3 \) is affinely independent if they are skew lines.

Prior Work

- Data from multiple affine subspaces is self-expressive, i.e., \( x_j = X c_j, X = [x_1, \ldots, x_N] \)
  - find the self-expression \( c_j \) via solving
    \[
    \min_{c_j} \quad \text{s.t.} \quad x_j = X c_j, \quad c_j = 0, \quad c_j \in C
    \]
  - apply spectral clustering to affinity \( |c_{ij}| + |c_{ji}| \)
- \( \text{ASC without an affine constraint: } C = \mathbb{R}^N \)
  - No explicit modeling of the affine structure
  - More often applied, many scalable algorithms
- \( \text{ASC with an affine constraint: } C = \{1 \cdot c_1 = 1\} \)
  - With explicit modeling of the affine structure
  - Rarely applied, no scalable algorithms (noisy case)

Contributions

- When the ambient dimension is high enough, the affine constraint is not needed.
- Randomly generated subspaces: both ASC with/without an affine constraint are guaranteed to produce correct affinity.
- Computer vision datasets: difference in performance between ASC with/without an affine constraint is small or negligible.

Empirical Evaluation for Affine Subspace Clustering

- We study the following methods:
  - Affine constraint: \( f() = \| \cdot \|_1 \)
  - With: A-SSC \quad A-LSR
- We test on the following real datasets:
  - Hopkins 155, MNIST, Coil-100

Conclusion:

For real applications, difference between ASC with/without an affine constraint is small for high-dimensional data.

Theoretical Analysis for Affine Subspace Clustering

Preliminaries

- **Definition:** A function \( f : \mathbb{R}^{N \times D} \rightarrow \mathbb{R} \) is said to satisfy the Enforced Block Diagonal (EBD) conditions if \( f(C) = f(P^T C P) \) for any permutation \( P \) and \( f(C) \geq f(C_0) \) for any \( C_0 \) that contains only the diagonal blocks of \( C \).
  - The EBD conditions are satisfied for \( f(\cdot) = \| \cdot \|_1, \| \cdot \|_2, \| \cdot \|_\infty \), and so on.
- **Definition:** A collection of affine subspaces \( \{A_1\}_{j=1}^N \) is said to be affinely independent if \( \dim(\text{aff}(\cup_{j=1}^N A_j)) = 1 + \sum_{j=1}^N \dim(A_j) + n \)
- **Definition:** A collection of affine subspaces \( \{A_1 \subseteq \mathbb{R}^D\}_{j=1}^N \) is said to be drawn from the random affine subspace model if they are drawn independently and uniformly from the set of affine \( (d_j) \)-dimensional subspaces of \( \mathbb{R}^D \).

Geometric Conditions

Given data \( x_i \in \mathbb{R}^D \) drawn from \( \{A_1\}_{j=1}^N \), assume that \( f \) satisfies the EBD conditions.

- **Theorem:** (ASC without affine constraint)
  - The solution to (1) gives a correct affinity if
    \( C_1: \{A_j\}_{j=1}^N \) is affinely independent
  - **C1:** \( \{A_j\}_{j=1}^N \) is affinely independent
- **Theorem:** (ASC with affine constraint)
  - The solution to (1) gives a correct affinity if
    \( C_2: 0 \not\in \text{aff}(\cup_{j=1}^N A_j) \)
  - **C2:** \( 0 \not\in \text{aff}(\cup_{j=1}^N A_j) \)

Conditions Under Random Affine Subspace Model

- **Theorem:** Let \( \{A_1 \subseteq \mathbb{R}^D\}_{j=1}^N \) be drawn from the random affine subspace model.

  \[
  \begin{align*}
  \Pr(C_1 \text{ is satisfied}) & \geq D \\\text{ and } \Pr(C_2 \text{ is satisfied}) \geq D \\
  \text{(For n = 4 and } D \text{ for n = 5 and } D_j = 4) \\
  \end{align*}
  \]

Conclusion:

For affine subspaces drawn from the random model, both ASC with/without an affine constraint produce correct affinities with probability 1 if \( D \geq n + \sum_{j=1}^N d_j \).

This work was supported by NSF under grant #1618637 and #1704458, Northrop Grumman Corporation’s REALM Program, NSFC under grant #61876022 and MOE at Peking University’s Open Project Fund.