Geometric Conditions for Subspace Sparse Recovery

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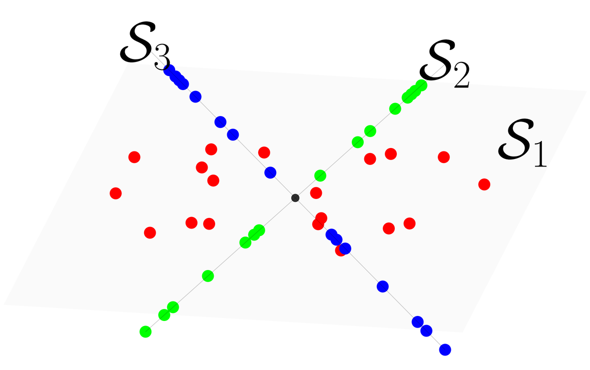
Motivations

- **Sparse recovery**: Find the unique sparsest representation $\mathbf{x}$ of a vector $\mathbf{b}$ with respect to a dictionary $\mathcal{A}$:

$$\min_{\mathbf{x}} \|\mathbf{x}\|_0 \quad \text{s.t.} \quad \mathbf{b} = \sum_j \mathbf{a}_j x_j. \quad (1)$$



- **Subspace sparse recovery**: Find a subspace sparse representation $\mathbf{x}$ of $\mathbf{b}$ whose support gives its membership to one or more low-dimensional subspaces.
- **Applications**: in machine learning, computer vision and signal processing.

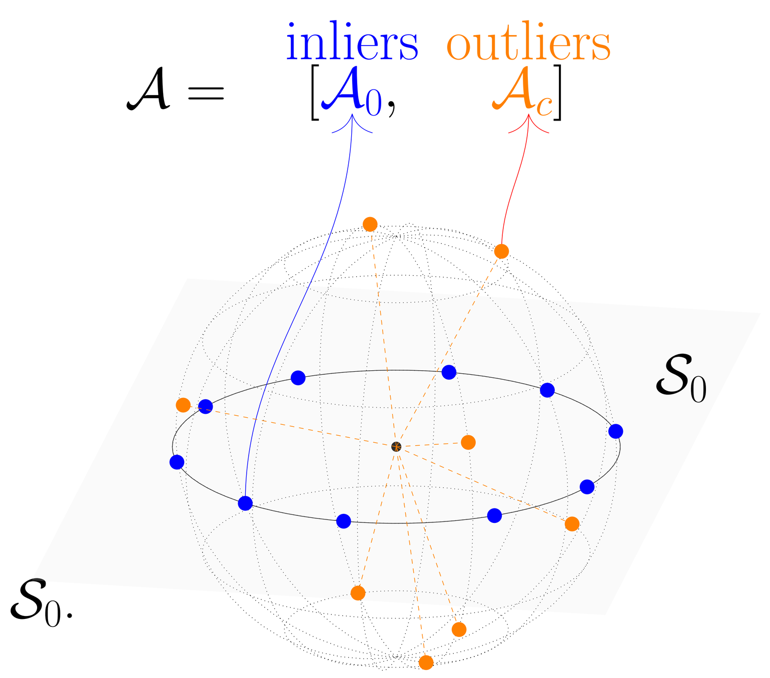
	Classification	Clustering
$\mathbf{b}$	Test data point	Data point
$\mathbf{a}_j$	All labeled data points	All other data points
$\mathbf{x}$	Gives membership	Gives pairwise affinity
Application	Face recognition	Face clustering

Problem Statement

- **Subspace**: $\mathcal{S}_0 \subset \mathbb{R}^D$.
- **Subspace structured dictionary**:

$\mathcal{A} = \{\mathbf{a}_j \in \mathbb{R}^D, j \in J\} = \mathcal{A}_0 \cup \mathcal{A}_c$,

where $\begin{cases} \mathcal{A}_0 \subseteq \mathcal{S}_0 : & \text{inliers to } \mathcal{S}_0 \\ \mathcal{A}_c \cap \mathcal{S}_0 = \emptyset : & \text{outliers to } \mathcal{S}_0 \end{cases}$



- **Subspace sparse representation**:

Let $\mathbf{b} \in \mathcal{S}_0$. Then $\mathbf{x}_j \neq 0 \implies \mathbf{a}_j \in \mathcal{A}_0 \subseteq \mathcal{S}_0$.

- **Problem**: Find conditions under which orthogonal matching pursuit (OMP) or basis pursuit (BP) give a subspace sparse representation for all $\mathbf{b} \in \mathcal{S}_0$.

Challenges

- **Sparse recovery**: dictionary needs to have low mutual coherence or satisfy the restricted isometry property so that the sparsest solution be unique.
- **Subspace sparse recovery**:
  - Points in a subspace can be arbitrarily close, hence $\mathcal{A}$ may not be incoherent.
  - Subspace sparse representations may not and need not be unique.

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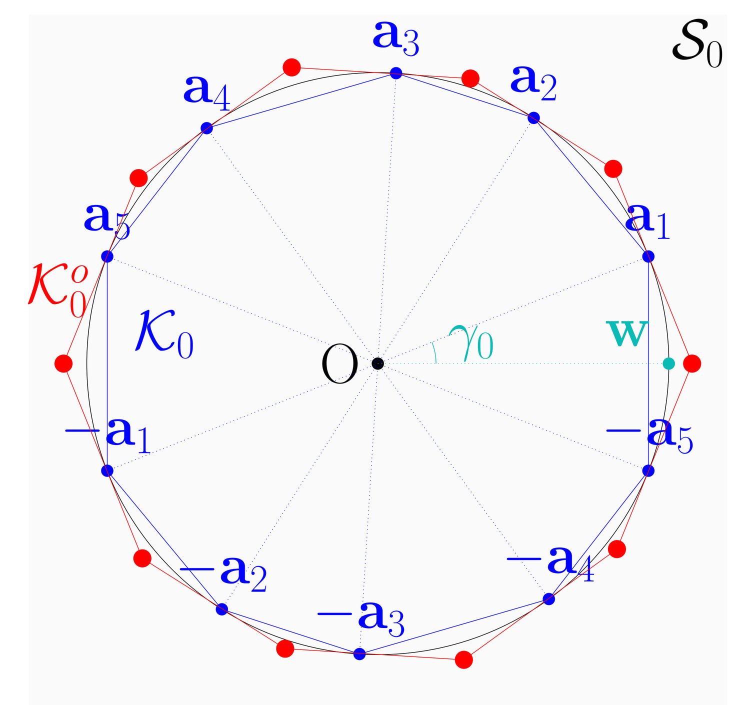
Main Result: Geometric Conditions

- **Theorem**: The solutions of OMP and BP are subspace sparse for all $\mathbf{b} \in \mathcal{S}_0$ if the dictionary $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_c$ satisfies either of the following conditions:

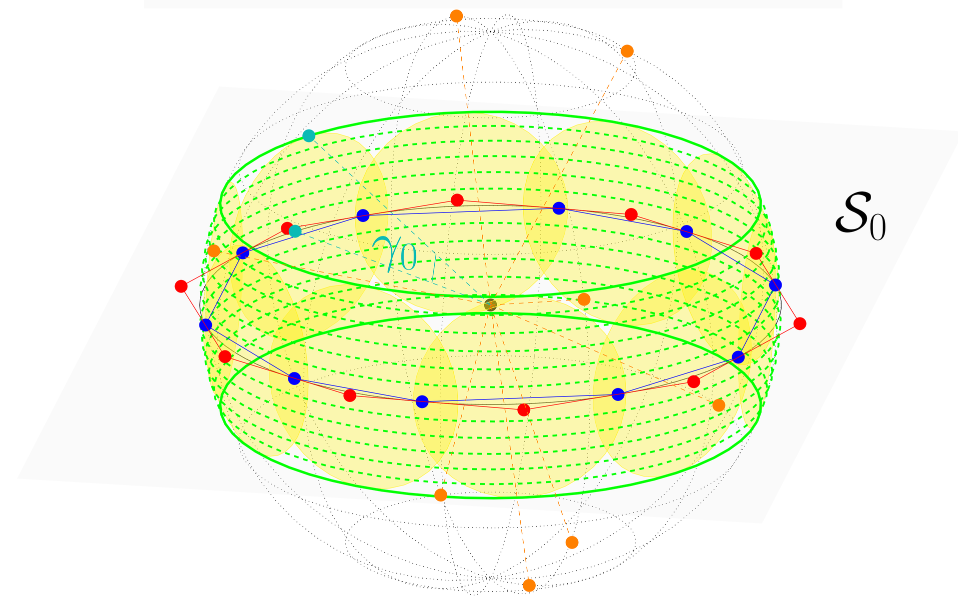
  Principal Recovery Condition (PRC): $\gamma_0 < s(\mathcal{A}_c, \mathcal{S}_0)$,
  Dual Recovery Condition (DRC): $\gamma_0 < s(\mathcal{A}_c, \mathcal{D}_0)$.



- **Notation**:
  - $s(\mathcal{A}_c, \mathcal{S}_0)$: minimum angle between the outliers $\mathcal{A}_c$ and the subspace $\mathcal{S}_0$, i.e.,

$$s(\mathbf{v}, \mathbf{w}) := \cos^{-1}\langle \frac{\mathbf{v}}{\|\mathbf{v}\|_2}, \frac{\mathbf{w}}{\|\mathbf{w}\|_2} \rangle, \; s(\mathcal{V}, \mathcal{W}) := \inf_{\mathbf{v} \in \mathcal{V} \setminus \{\mathbf{0}\}} \inf_{\mathbf{w} \in \mathcal{W} \setminus \{\mathbf{0}\}} s(\mathbf{v}, \mathbf{w}).$$

  Note that s is larger if points from two sets are not close.
  - $s(\mathcal{A}_c, \mathcal{D}_0)$: minimum angle between $\mathcal{A}_c$ and $\mathcal{D}_0$, where $\mathcal{D}_0$ = the set of extreme points of $\{\mathbf{v} \in \mathcal{S}_0 : \langle \mathbf{v}, \mathbf{w} \rangle \leq 1, \forall \mathbf{w} \in \operatorname{conv}\{\pm \mathcal{A}_0\}\}$ is a finite subset of $\mathcal{S}_0$.
  - $\gamma_0 = \max\{s(\pm \mathcal{A}_0, \mathbf{w}) : \mathbf{w} \in \mathcal{S}_0\}$: covering radius of points $\pm \mathcal{A}_0$ in $\mathcal{S}_0$. Note that γ_0 is smaller if the unit sphere of the subspace $\mathcal{S}_0$ is densely populated with points $\pm \mathcal{A}_0$.



- **Geometric interpretation**:
  - **PRC** holds if and only if all points in $\mathcal{A}_c$ lie outside the green region.
  - **DRC** holds if and only if all points in $\mathcal{A}_c$ lie outside the yellow region.

Implications for Sparse and Subspace Sparse Recovery

Implications for subspace sparse recovery

- To achieve subspace sparse recovery, the inliers $\mathcal{A}_0$ should be dense and well distributed in subspace $\mathcal{S}_0$ so that γ_0 is small, and the outliers $\mathcal{A}_c$ should be sufficiently far from the subspace $\mathcal{S}_0$ (by PRC) or the set $\mathcal{D}_0$ (by DRC).
- DRC is a weaker condition than PRC since it only requires a finite subset of $\mathcal{S}_0$ to be far from the outliers.
- PRC and DRC justify why subspace sparsity works better for low-dimensional subspaces, since the area of the PRC/DRC region relative to the area of the entire unit sphere is smaller.

Implications for sparse recovery

- **Theorem**: Both OMP and BP recover any s-sparse signal with dictionary $\mathcal{A}$ if $\forall \mathcal{A}_0 \subseteq \mathcal{A}$ s.t. $\operatorname{card}(\mathcal{A}_0) = s$ we have $\operatorname{rank}(\mathcal{A}_0) = s$ and either PRC or DRC hold.
- **Properties**:
  - Geometrically interpretable: requires points to be "incoherent".
  - Can be verified: explicit formula for computing $\mathcal{D}_0$ and γ_0.
  - Less restrictive: the condition in theorem is implied by the mutual coherent condition $\mu(\mathcal{A}) < \frac{1}{2s-1}$, where $\mu(\mathcal{A}) = \max_{j \neq k} |\langle \mathbf{a}_j, \mathbf{a}_k \rangle|$.

References

[1] C. You and R. Vidal., Subspace Sparse Representation, In *Arxiv*, 2015.

[2] M. Soltanolkotabi and E.J. Candes., A geometric analysis of subspace clustering with outlier, In *Annals of Statistics*, 2013.