Geometric Conditions for Subspace Sparse Recovery

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Motivations

- **Sparse recovery**: Find the unique sparsest representation \( \mathbf{x} \) of a vector \( \mathbf{b} \) with respect to a dictionary \( \mathcal{A} \):

\[
\min_{\mathbf{x}} \| \mathbf{x} \|_0 \quad \text{s.t.} \quad \mathbf{b} = \sum_j x_j \mathbf{a}_j.
\]

- **Subspace sparse recovery**: Find a subspace sparse representation \( \mathbf{x} \) of \( \mathbf{b} \) whose support gives its membership to one or more low-dimensional subspaces.

**Application**: in machine learning, computer vision and signal processing.

<table>
<thead>
<tr>
<th>Classification</th>
<th>Clustering</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{b} )</td>
<td>Test data point</td>
</tr>
<tr>
<td>( \mathbf{a}_j )</td>
<td>All labeled data points</td>
</tr>
<tr>
<td>( \mathbf{x} )</td>
<td>Gives membership</td>
</tr>
</tbody>
</table>

| Application | Face recognition | Face clustering |

Problem Statement

- **Subspace**: \( S_0 \subset \mathbb{R}^D \).
- **Subspace structured dictionary**: \( \mathcal{A} = \{ \mathbf{a}_j \in \mathbb{R}^D, j \in J \} = \mathcal{A}_0 \cup \mathcal{A}_s \),

\[
\begin{aligned}
\mathcal{A}_0 &\subseteq S_0 : \text{inliers to } S_0 \\
\mathcal{A}_s &\cap S_0 = \emptyset : \text{outliers to } S_0
\end{aligned}
\]

- **Subspace sparse representation**: Let \( \mathbf{b} \in S_0 \). Then \( x_j \neq 0 \Rightarrow \mathbf{a}_j \in \mathcal{A}_0 \subseteq S_0 \).

- **Problem**: Find conditions under which orthogonal matching pursuit (OMP) or basis pursuit (BP) give a subspace sparse representation for all \( \mathbf{b} \in S_0 \).

Main Result: Geometric Conditions

**Theorem**: The solutions of OMP and BP are subspace sparse for all \( \mathbf{b} \in S_0 \) if the dictionary \( \mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_s \) satisfies either of the following conditions:

- **Principal Recovery Condition (PRC)**: \( \gamma_0 < s(\mathcal{A}_s, S_0) \),
- **Dual Recovery Condition (DRC)**: \( \gamma_0 < s(\mathcal{A}_s, D_0) \).

**Notation**:
- \( s(\mathcal{A} \setminus \mathcal{A}_s, S_0) \): minimum angle between the outliers \( \mathcal{A}_s \) and the subspace \( S_0 \), i.e.,

\[
s(\mathbf{v}, \mathbf{w}) := \cos^{-1} \left( \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\| \mathbf{v} \|_2 \| \mathbf{w} \|_2} \right),
\]

\[
s(\mathcal{V}, \mathcal{W}) := \inf_{\mathbf{v} \in \mathcal{V}(0)} \inf_{\mathbf{w} \in \mathcal{W}(0)} s(\mathbf{v}, \mathbf{w}).
\]

Note that \( s \) is larger if points from two sets are not close.

**Geometric interpretation**:
- PRC holds if and only if all points in \( \mathcal{A}_s \) lie outside the green region.
- DRC holds if and only if all points in \( \mathcal{A}_s \) lie outside the yellow region.

Implications for Sparse and Subspace Sparse Recovery

**Implications for subspace sparse recovery**

- To achieve subspace sparse recovery, the inliers \( \mathcal{A}_0 \) should be dense and well distributed in subspace \( S_0 \) so that \( \gamma_0 \) is small, and the outliers \( \mathcal{A}_s \) should be sufficiently far from the subspace \( S_0 \) (by PRC) or the set \( D_0 \) (by DRC).

**Implications for sparse recovery**

- **Theorem**: Both OMP and BP recover any \( s \)-sparse signal with dictionary \( \mathcal{A} \) if \( \mathcal{A} \subseteq \mathcal{A} \) s.t. \( \text{card}(\mathcal{A}_0) = s \) we have \( \text{rank}(\mathcal{A}_0) = s \) and either PRC or DRC hold.

**Properties**:
- Geometrically interpretable: requires points to be “incoherent”.
- Can be verified: explicit formula for computing \( D_0 \) and \( \gamma_0 \).
- Less restrictive: the condition in theorem is implied by the mutual coherence condition \( \mu(\mathcal{A}) < \frac{1}{\sqrt{m}} \) where \( \mu(\mathcal{A}) = \max_j \mu_j(\mathcal{A}_j) \).

References


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