Provably Convergent On-line Structure and Motion Estimation for Perspective Systems *

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Abstract

Estimation of structure and motion in computer vision systems can be performed using a dynamic systems approach, where states and parameters in a perspective system are estimated. This paper presents a new approach to the structure estimation problem, where the estimation of the 3D-positions of feature points on a moving object is reformulated as a parameter estimation problem. For each feature point, a constant parameter is estimated, from which it is possible to calculate the time-varying 3D-position. The estimation method is extended to the estimation of motion, in the form of angular velocity estimation. The combined structure and angular velocity estimator is shown stable using Lyapunov theory and persistency of excitation based arguments. The estimation method is illustrated with simulation examples, demonstrating the estimation convergence.

1. Introduction

Estimation of 3D information from 2D images in computer vision systems can be performed using dynamic systems. Estimation of 3D *positions* of observed feature points in a sequence of images is often referred to as *structure estimation*, and estimation of the corresponding motion parameters, e.g. expressed using angular and linear velocities, is often referred to as *motion estimation*.

A specific class of algorithms for structure estimation, where available values for the angular and linear velocities are used and where position is estimated, can be formulated as *nonlinear observers*. This kind of algorithms are described e.g. in [18, 11, 19, 2, 9, 7, 1, 15, 12, 10, 5, 8], where estimators for *structure only* are presented, based on different nonlinear observers. Various analytical results regarding stability are provided in the mentioned references, and simulation examples are used to illustrate the performance of the different observers.

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This paper describes how a parametrization of the underlying perspective dynamic system can be used to formulate the structure estimation problem as an estimation problem, where the task is to estimate a *constant* parameter. The parametrization is derived using similar methodology as used in an earlier introduced parametrization [5], and can be regarded as a reformulation of the estimation problem, resulting in a simplified analysis due to the constant parameter estimation approach.

The parametrization allows for estimation of structure as well as motion, which is the case also for the parametrization in [5], as shown e.g. in [6, 4]. The stability of the structure estimation problem can be analyzed using Lyapunov theory and persistency of excitation based reasoning. This is the case also for the parametrization in [5], which was analyzed with respect to stability in [4]. The stability analysis for the new parametrization, as presented below, is however simplified compared to the analysis in [4], due to the different parametrization.

The perspective dynamic system, which is the base for derivation of estimators for structure and motion, is reviewed in Section 2, and the earlier parametrization method introduced in [5] is reviewed in Section 3.

The main results are given in Section 4 and Section 5 where algorithms are given, and in Section 6 where the stability of the structure and angular velocity estimator is analyzed. Simulation examples are presented in Section 7.

2. Perspective Dynamic System

Consider a moving object observed by a camera. The object is rigid, and it has N feature points. The feature points are extracted by image processing and then used as input to the 3D-reconstruction estimation. The three-dimensional coordinates of the feature points are denoted

$$x^i = \begin{pmatrix} x_{1,i} & x_{2,i} & x_{3,i} \end{pmatrix}^T, \quad i = 1, \ldots, N \tag{1}$$

The angular velocity of the object is denoted

$$\omega = \begin{pmatrix} \omega_1 & \omega_2 & \omega_3 \end{pmatrix}^T \tag{2}$$

The translational velocity is given by a vector denoted

$$b = \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix}^T \tag{3}$$

Introduce the quantity

$$\xi^i = \begin{pmatrix} \frac{x_{1,i}}{x_{3,i}} & \frac{x_{2,i}}{x_{3,i}} \end{pmatrix}^T, \quad i = 1, \ldots, N \tag{4}$$

and the camera parameters $C_f \in \mathbb{R}^{2\times 2}$ and $\delta \in \mathbb{R}^{2\times 1}$. A perspective camera (e.g. [16]) then computes the image coordinates

$$y^i = \begin{pmatrix} y_{1,i} & y_{2,i} \end{pmatrix}^T, \quad i = 1, \ldots, N \tag{5}$$

as

$$y^i = C_f \xi^i + \delta \tag{6}$$

Introducing a skew-symmetric matrix A, defined using the angular velocity (2) as

$$A = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \tag{7}$$

the motion of the rigid body object, together with its observed images, is then described by the perspective dynamic system

$$\begin{aligned} \dot{x}^i &= Ax^i + b \\ y^i &= C_f \xi^i + \delta \end{aligned} \quad , \quad i \in \{1, 2 \ldots N\} \ . \tag{8}$$

Note that the quantities A, b, C_f and δ, as a result of the rigid body assumption and the use of a single camera, are common to all the points x^i. When considering only one point, the simplified notation

$$\begin{aligned} \dot{x} &= Ax + b \\ y &= C_f \xi + \delta \end{aligned} \tag{9}$$

will be used.

Introduce a mapping, denoted S, which given a vector

$$v = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}^T \tag{10}$$

defines a skew-symmetric matrix according to

$$S(v) = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} \tag{11}$$

The matrix A is then defined from the angular velocity as

$$A = S(\omega) \tag{12}$$

3. Dynamic Vision Parametrization

Considering one point, i.e. using the simplified notation (9), a dynamic vision parametrization, introduced in [5], can be derived. For this purpose, introduce the scalar parameter γ and the vector z by

$$\gamma = \frac{1}{\sqrt{x^T x}}, \quad z = \gamma x \tag{13}$$

Observe that ξ, according to (4) and by the definition of z in (13), also can be expressed as

$$\xi = \begin{pmatrix} \frac{z_1}{z_3} & \frac{z_2}{z_3} \end{pmatrix}^T \ , \tag{14}$$

From the equation for y in (9) it can be seen that if the camera is calibrated, i.e. if C_f and δ in (9) are known, and further if the matrix C_f is invertible, then ξ can be calculated given y (and hence assumed known). This means, using (13) and the definition of ξ in (4), that the vector z which then can be expressed as

$$z = \frac{1}{\sqrt{\xi_1^2 + \xi_2^2 + 1}} \begin{pmatrix} \xi_1 & \xi_2 & 1 \end{pmatrix}^T \tag{15}$$

can also be assumed known.

Hence, z is a measurable signal, and can therefore be considered an output of the system (9). The dynamic vision parametrization exploits this fact, and aims at rewriting the system (9) so that z appears explicitly in the equations.

Using (9) and the fact that $x^T Ax = 0$, since A is skew-symmetric, and introducing

$$g_0(z) = I - zz^T \tag{16}$$

gives a rewritten dynamic system, corresponding to (9), on the form

$$\begin{aligned} \dot{z} &= Az + g_0(z)b\gamma \\ \dot{\gamma} &= -\gamma^2 z^T b \ . \end{aligned} \tag{17}$$

For the motion of more than one point a dynamic system corresponding to (8) is obtained as

$$\begin{aligned} \dot{z}^i &= Az^i + g_0(z^i)b\gamma^i \\ \dot{\gamma}^i &= -(\gamma^i)^2 (z^i)^T b \end{aligned} \quad , \quad i \in \{1, 2 \ldots N\} \ . \tag{18}$$

Equation (13) together with (17) and its multipoint version (18), constitute the desired dynamic vision parametrization, from which we shall proceed.

4. Structure Estimation as Parameter Estimation

4.1. Reformulation of the dynamic vision parametrization

Introduce a quantity α, defined as

$$\alpha = \frac{1}{\gamma} \tag{19}$$

with time derivative $\dot{\alpha}$, computed using the last equation in (17), as

$$\dot{\alpha} = -\frac{1}{\gamma^2}\dot{\gamma} = -\frac{1}{\gamma^2}(-\gamma^2 z^T b) = z^T b \tag{20}$$

An alternative dynamic vision parametrization can then be obtained, by rewriting (17) as

$$\begin{aligned} \dot{z} &= Az + g_0(z)b\frac{1}{\alpha} \\ \dot{\alpha} &= z^T b \end{aligned} \tag{21}$$

Using

$$\int_{t_0}^{t} \dot{\alpha}(\tau)d\tau = \alpha(t) - \alpha(t_0) \tag{22}$$

and

$$\int_{t_0}^{t} \dot{\alpha}(\tau)d\tau = \int_{t_0}^{t} z^T(\tau)b(\tau)d\tau \tag{23}$$

we get

$$\alpha(t) = \alpha(t_0) + \int_{t_0}^{t} z^T(\tau)b(\tau)d\tau \tag{24}$$

Introducing

$$\alpha_0 = \alpha(t_0) \tag{25}$$

and

$$\zeta(t) = \int_{t_0}^{t} z^T(\tau)b(\tau)d\tau \tag{26}$$

where we note that $\zeta(t)$ is a measurable quantity, an expression for $\alpha(t)$ is obtained as

$$\alpha(t) = \alpha_0 + \zeta(t) \tag{27}$$

The dynamic system (21) can now be written as

$$\dot{z} = Az + g_0(z)b\frac{1}{\alpha_0 + \zeta} \tag{28}$$

where the differential equation $\dot{\alpha}_0 = 0$ is omitted. The constant quantity α_0 can thus be interpreted as a *parameter* in (28), where all quantities except α_0 are known.

4.2. Structure estimation algorithm

A structure estimation formulation can be formulated as the task of estimating α_0 in (28). Given knowledge of α_0, an estimate of the object position can be obtained, using the equation for z in (13) together with (19), (25), (26) and (27), to obtain an equation for x, as

$$x = z \cdot (\alpha_0 + \zeta) \tag{29}$$

An estimator for α_0 can be designed as follows. Introduce a matrix F which is Hurwitz, and a symmetric positive definite matrix Q. A symmetric positive definite matrix P can then be computed as the unique solution to the Lyapunov equation [13],

$$F^T P + PF = -Q \ . \tag{30}$$

Introducing the estimated quantities $\hat{z}$ and $\hat{\alpha}_0$, an estimator can be formulated as

$$\begin{aligned} \dot{\hat{z}} &= Az + F(\hat{z} - z) + g_0(z)b\frac{1}{\hat{\alpha}_0 + \zeta} \\ \dot{\hat{\alpha}}_0 &= b^T g_0(z)^T P(\hat{z} - z) \ . \end{aligned} \tag{31}$$

The estimator (31) is inspired by [20], where however the parameters to be estimated appears linearly. Even if this is not the case for α_0 in (28), we choose the update law for α_0 as proposed in [20].

Defining the estimation errors

$$\tilde{z} = \hat{z} - z, \quad \tilde{\alpha}_0 = \hat{\alpha}_0 - \alpha_0 \ , \tag{32}$$

while also observing that $\dot{\alpha}_0 = 0$, a system of error differential equations can be obtained by combining (28) and (31), as

$$\begin{aligned} \dot{\tilde{z}} &= F\tilde{z} + g_0(z)b\left(\frac{1}{\hat{\alpha}_0 + \zeta} - \frac{1}{\alpha_0 + \zeta}\right) \\ \dot{\tilde{\alpha}}_0 &= b^T g_0(z)^T P\tilde{z} \ . \end{aligned} \tag{33}$$

The term

$$\frac{1}{\hat{\alpha}_0 + \zeta} - \frac{1}{\alpha_0 + \zeta} \tag{34}$$

can be rewritten as

$$\frac{1}{\hat{\alpha}_0 + \zeta} - \frac{1}{\alpha_0 + \zeta} = \frac{\alpha_0 + \zeta - \hat{\alpha}_0 - \zeta}{(\hat{\alpha}_0 + \zeta)(\alpha_0 + \zeta)} \tag{35}$$

which, when using the definition of $\tilde{\alpha}_0$ in (32) can be regarded as a function of $\tilde{\alpha}_0$, here denoted $\sigma(\tilde{\alpha}_0)$, as

$$\sigma(\tilde{\alpha}_0) = \frac{-\tilde{\alpha}_0}{(\tilde{\alpha}_0 + \alpha_0 + \zeta)(\alpha_0 + \zeta)} \tag{36}$$

Using (36), the estimator error equations (33) can be written as

$$\begin{aligned} \dot{\tilde{z}} &= F\tilde{z} + g_0(z)b\sigma(\tilde{\alpha}_0) \\ \dot{\tilde{\alpha}}_0 &= b^T g_0(z)^T P\tilde{z} \ . \end{aligned} \tag{37}$$

The estimator (31) is considered asymptotically stable if the estimation errors $\tilde{z}$ and $\tilde{\alpha}_0$ approach zero. An estimate of the 3D-coordinates x can then be obtained, using (29), as

$$\hat{x} = z(\hat{\alpha}_0 + \zeta) \tag{38}$$

An alternative formulation could be to use

$$\hat{x} = \hat{z}(\hat{\alpha}_0 + \zeta) \tag{39}$$

instead, where the estimated value of z is used instead of the measured value. The purpose of this approach is to obtain an increased noise sensitivity, which would be the case provided the estimator (31) has a low-pass filtering effect on the measured signal z. It can also be observed that the convergence rate of the estimator is affected by the choice of the matrices F and Q, which therefore could be considered as design parameters for the estimator (31).

5. EXTENSION TO ANGULAR VELOCITY ESTIMATION

As can be seen in (28), the parameter A appears linearly. This could be used for development of estimators for the angular velocity ω, which is related to the skew-symmetric matrix A according to (12). Since multiplication of a vector $v \in \mathbb{R}^{3\times 1}$ by a skew-symmetric matrix can be equivalently formulated as a vector cross product, i.e. $S(\omega)v = \omega \times v$, using $\omega \times z = -z \times \omega$ then gives $S(\omega)z = -S(z)\omega$. The system (28) can therefore be rewritten as

$$\dot{z} = -S(z)\omega + g_0(z)b\frac{1}{\alpha_0 + \zeta} \tag{40}$$

Considering the estimation problem for the case of a constant ω, again using inspiration from [20], the estimator (31) can be extended to an estimator where also the angular velocity ω is estimated, as

$$\begin{aligned}
\dot{\hat{z}} &= -S(z)\hat{\omega} + F(\hat{z} - z) + g_0(z)b\frac{1}{\hat{\alpha}_0 + \zeta} \\
\dot{\hat{\alpha}}_0 &= b^T g_0(z)^T P(\hat{z} - z) \\
\dot{\hat{\omega}} &= \hat{\alpha}^2 S(z)^T P(\hat{z} - z)\ \ ,
\end{aligned} \tag{41}$$

where the factor $\hat{\alpha}^2$ in the last equation is needed in order to prove stability later on. It can be regarded as a weighting/time scaling component in the estimation of the angular velocity. Defining the angular velocity estimation error

$$\tilde{\omega} = \hat{\omega} - \omega \tag{42}$$

and using (32), a system of error differential equations can be obtained by combining (40) and (41), as

$$\begin{aligned}
\dot{\tilde{z}} &= F\tilde{z} + g_0(z)b\left(\frac{1}{\hat{\alpha}_0 + \zeta} - \frac{1}{\alpha_0 + \zeta}\right) - S(z)\tilde{\omega} \\
\dot{\tilde{\alpha}}_0 &= b^T g_0(z)^T P\tilde{z} \\
\dot{\tilde{\omega}} &= \hat{\alpha}^2 S(z)^T P\tilde{z}
\end{aligned} \tag{43}$$

6. Stability Analysis

A stability analysis for the case of structure estimation is presented. The analysis shows asymptotic stability of the estimator, hence giving a proof of convergence in the sense that if the initial structure estimate is close enough to its true value, the estimate will converge as the time $t \to \infty$.

The purpose of the stability analysis is to show that the estimation errors $\tilde{z}$, $\tilde{\alpha}_0$ and $\tilde{\omega}$ in (43) approach zero.

We will show stability of the error equations by Lyapunov's indirect method, where stability of a nonlinear system is deduced from the stability of a linear system, obtained from a *linearization* of the nonlinear system [13].

Linearization of (43) results in

$$\begin{aligned}
\dot{\tilde{z}} &= F\tilde{z} - g_0(z)b\frac{1}{(\alpha_0 + \zeta)^2}\tilde{\alpha}_0 - S(z)\tilde{\omega} \\
\dot{\tilde{\alpha}}_0 &= b^T g_0(z)^T P\tilde{z} \\
\dot{\tilde{\omega}} &= \alpha^2 S(z)^T P\tilde{z}
\end{aligned} \tag{44}$$

Using (27), equation (44) is rewritten as

$$\begin{aligned}
\dot{\tilde{z}} &= F\tilde{z} - g_0(z)b\frac{1}{\alpha^2}\tilde{\alpha}_0 - S(z)\tilde{\omega} \\
\dot{\tilde{\alpha}}_0 &= b^T g_0(z)^T P\tilde{z} \\
\dot{\tilde{\omega}} &= \alpha^2 S(z)^T P\tilde{z}
\end{aligned} \tag{45}$$

As a first step, we show that $\tilde{z} \to 0$ and that $\tilde{\alpha}_0$ and $\tilde{\omega}$ are bounded. Introduce the Lyapunov function

$$V(\tilde{z}, \tilde{\alpha}_0, \tilde{\omega}) = \alpha^2\frac{1}{2}\tilde{z}^T P\tilde{z} + \frac{1}{2}\tilde{\alpha}_0^2 + \frac{1}{2}\tilde{\omega}^T\tilde{\omega} \tag{46}$$

Differentiating (46) along the trajectories of (45), gives (assuming F is symmetric for simplicity, but the final result holds for general F)

$$\begin{aligned}
&\dot{V}(\tilde{z}, \tilde{\alpha}_0, \tilde{\omega}) = \\
&= \alpha\dot{\alpha}\tilde{z}^T P\tilde{z} + \alpha^2\tilde{z}^T P\left(F\tilde{z} - g_0(z)b\frac{1}{\alpha^2}\tilde{\alpha}_0 - S(z)\tilde{\omega}\right) \\
&\quad + \tilde{\alpha}_0 b^T g_0(z)^T P\tilde{z} + \alpha^2\tilde{\omega}^T S(z)^T P\tilde{z} \\
&= \tilde{z}^T\alpha\dot{\alpha}P\tilde{z} + \alpha^2\tilde{z}^T PF\tilde{z} - \tilde{z}^T P g_0(z)b\tilde{\alpha}_0 \\
&\quad - \alpha^2\tilde{z}^T PS(z)\tilde{\omega} + \tilde{\alpha}_0 b^T g_0(z)^T P\tilde{z} + \alpha^2\tilde{\omega}^T S(z)^T P\tilde{z} \\
&= \tilde{z}^T\alpha\dot{\alpha}P\tilde{z} + \alpha^2\tilde{z}^T PF\tilde{z}\ \ .
\end{aligned} \tag{47}$$

Using the Lyapunov equation (30), equation (47) can be written as

$$\begin{aligned}
\dot{V}(\tilde{z}, \tilde{\alpha}_0, \tilde{\omega}) &= -\frac{1}{2}\alpha^2\tilde{z}^T Q\tilde{z} + \tilde{z}^T\alpha\dot{\alpha}P\tilde{z} \\
&= \frac{1}{2}\alpha^2\tilde{z}^T(-Q + 2\frac{\dot{\alpha}}{\alpha}P)\tilde{z}
\end{aligned} \tag{48}$$

Assuming that the matrices Q and P are such that the matrix

$$M = -Q + 2\frac{\dot{\alpha}}{\alpha}P \tag{49}$$

is negative definite for the object motions considered, i.e. for the range of values for z, b, ζ and α_0, we get

$\dot{V}(\tilde{z}, \tilde{\alpha}_0, \tilde{\omega}) \leq 0$ from which we can conclude that $\tilde{z} \to 0$ and that $\tilde{\alpha}_0$ and $\tilde{\omega}$ are bounded, e.g. [13, 17].

The matrix M in (49) can be rewritten as

$$M = -Q + 2\frac{d}{dt}\ln(\alpha)P \tag{50}$$

Observing that α is the distance from the camera to the object, the matrix M is thus a function of the logarithm of the distance. Hence, with some information about the possible motions, i.e. how fast $\ln(\alpha)$ can change, it should be possible to choose Q and P such that M is negative definite, while still fulfilling (30), e.g. by choosing the eigenvalues of Q sufficiently large.

In order to show that also $\tilde{\alpha}_0 \to 0$ and $\tilde{\omega} \to 0$ a reasoning inspired by a stability proof in [17] will be used. First, a function $\varphi(t)$ is defined as

$$\varphi(t) = \frac{1}{2}\left(\tilde{\alpha}_0(t+T)^2 - \tilde{\alpha}_0(t)^2 + \tilde{\omega}(t+T)^2 - \tilde{\omega}(t)^2\right) \tag{51}$$

The function $\varphi(t)$ is bounded since $\tilde{\alpha}_0$ and $\tilde{\omega}$ are bounded. The time derivative $\dot{\varphi}(t)$ becomes

$$\begin{aligned} \dot{\varphi}(t) &= \tilde{\alpha}_0(t+T)\dot{\tilde{\alpha}}_0(t+T) - \tilde{\alpha}_0(t)\dot{\tilde{\alpha}}_0(t) \\ &\quad + \tilde{\omega}(t+T)^T\dot{\tilde{\omega}}(t+T) - \tilde{\omega}(t)^T\dot{\tilde{\omega}}(t) \\ &= \int_t^{t+T} \frac{d}{d\tau}(\tilde{\alpha}_0(\tau)\dot{\tilde{\alpha}}_0(\tau) + \tilde{\omega}(\tau)^T\dot{\tilde{\omega}}(\tau))d\tau \end{aligned} \tag{52}$$

The integral in (52) can be rewritten, using (45), resulting in

$$\begin{aligned} \frac{d}{dt}(\tilde{\alpha}_0\dot{\tilde{\alpha}}_0 + \tilde{\omega}^T\dot{\tilde{\omega}}) &= \frac{d}{dt}(\tilde{\alpha}_0 b^T g_0(z)^T P\tilde{z} + \tilde{\omega}^T\alpha^2 S(z)^T P\tilde{z}) \\ &= \tilde{\alpha}_0 b^T g_0(z)^T P\dot{\tilde{z}} + \frac{d}{dt}(\tilde{\alpha}_0 b^T g_0(z)^T P)\tilde{z} \\ &\quad + \alpha^2\tilde{\omega}^T S(z)^T P\dot{\tilde{z}} + \frac{d}{dt}(\alpha^2\tilde{\omega}^T S(z)^T P)\tilde{z} \end{aligned} \tag{53}$$

Using the first equation in (45), we get

$$\begin{aligned} \frac{d}{dt}(\tilde{\alpha}_0\dot{\tilde{\alpha}}_0 + \tilde{\omega}^T\dot{\tilde{\omega}})) &= -\tilde{\alpha}_0 b^T g_0(z)^T P g_0(z) b\frac{1}{\alpha^2}\tilde{\alpha}_0 \\ &\quad + \tilde{\alpha}_0 b^T g_0(z)^T PF\tilde{z} + \frac{d}{dt}(\tilde{\alpha}_0 b^T g_0(z)^T P)\tilde{z} \\ &\quad + \alpha^2\tilde{\omega}^T S(z)^T PF\tilde{z} - \alpha^2\tilde{\omega}^T S(z)^T P g_0(z) b\frac{1}{\alpha^2}\tilde{\alpha}_0 \\ &\quad - \alpha^2\tilde{\omega}^T S(z)^T PS(z)\tilde{\omega} + \frac{d}{dt}((\alpha^2)\tilde{\omega}^T S^T P)\tilde{z} \ . \end{aligned} \tag{54}$$

Introducing the matrix

$$N = \begin{bmatrix} \frac{1}{\alpha(\tau)^2} b(\tau)^T g_0(z(\tau))^T P g_0(z(\tau)) b(\tau) & -b(\tau)^T g_0(z(\tau))^T PS(z(\tau)) \\ -S(z(\tau))^T P g_0(z(\tau)) b(\tau) & -\alpha(\tau)^2 S(z(\tau))^T PS(z(\tau)) \end{bmatrix},$$

the integral in (52) becomes

$$\begin{aligned} &\dot{\varphi}(t) = \\ &-\int_t^{t+T} \begin{bmatrix}\tilde{\alpha}_0 & \tilde{\omega}(\tau)^T\end{bmatrix} N \begin{bmatrix}\tilde{\alpha}_0 \\ \tilde{\omega}(\tau)\end{bmatrix} d\tau \\ &+\int_t^{t+T} \tilde{\alpha}_0(\tau)b(\tau)^T g_0(z(\tau))^T PF\tilde{z}(\tau)d\tau \\ &+\int_t^{t+T} \frac{d}{d\tau}(\tilde{\alpha}_0(\tau)b(\tau)^T g_0(z(\tau))^T P)\tilde{z}(\tau)d\tau \\ &+\int_t^{t+T} \alpha^2(\tau)\tilde{\omega}(\tau)^T S(z(\tau))^T PF\tilde{z}(\tau)d\tau \\ &+\int_t^{t+T} \frac{d}{dt}(\alpha(\tau)^2)\tilde{\omega}(\tau)^T S(z(\tau))^T\tilde{z}(\tau)d\tau \ . \end{aligned} \tag{55}$$

If we now assume a *persistency of excitation condition*, also denoted PE condition, i.e. that for all t and T, there is a positive number k such that

$$\int_t^{t+T} \begin{bmatrix}\tilde{\alpha}_0 & \tilde{\omega}(\tau)^T\end{bmatrix} N \begin{bmatrix}\tilde{\alpha}_0 \\ \tilde{\omega}(\tau)\end{bmatrix} d\tau \geq kI \ , \tag{56}$$

we can show that $\tilde{\alpha}_0 \to 0$ and $\tilde{\omega} \to 0$ by the following reasoning.

Assume, contradictory to what we would like to show, that it is not the case that $\tilde{\alpha}_0 \to 0$ and $\tilde{\omega} \to 0$, i.e. either $\tilde{\alpha}_0 \to 0$ is violated or $\tilde{\omega} \to 0$ is violated. The PE condition (56) then implies that the first integral in (55) fulfils

$$\int_t^{t+T} \begin{bmatrix}\tilde{\alpha}_0 & \tilde{\omega}(\tau)^T\end{bmatrix} N \begin{bmatrix}\tilde{\alpha}_0 \\ \tilde{\omega}(\tau)\end{bmatrix} d\tau > 0 \ . \tag{57}$$

We also see that, for t large enough, the first integral in (55) will dominate over the two remaining integrals, which all tend to zero since $\tilde{z} \to 0$. Hence it is possible to achieve, possibly by selecting Q such that the eigenvalues of P are sufficiently large (which can be done, according to e.g. [14]), that for t large enough, there exists a time t_1 for which it holds that $\dot{\varphi}(t) < 0, \quad \forall t > t_1$ which contradicts the fact that $\varphi(t)$ is bounded. Hence, $\tilde{\alpha}_0 \to 0$ and $\tilde{\omega} \to 0$.

To summarize, we have shown asymptotical stability of the error equations (37) by using the corresponding linearized equations (45), provided the matrix (49) is negative definite for the motions considered and also that the PE condition (56) holds.

The PE condition (56) can be interpreted in terms of unfavourable motions, using the following reasoning. From (16) and the observation that z by its definition (13) is a vector of unit length, it can be seen that if $b \parallel z$ over some time interval, the upper left part of the integrand of (56) will be identically zero over that interval, and the PE condition will not be fulfilled. Except for the case $b = 0$, this means that the translational velocity of the observed point relative to the camera should not be directed along a straight line connecting the camera center with the current measurement z during any time interval. The observation that $b \parallel z$ results in an unfavorable motion from an observability point of view can also be seen directly from (28), since such a b would disrupt the influence of α_0 on the z-dynamics, and thus render the parameter identification process infeasible, since then the parameter α_0 cannot be observed through the dynamics of the available signal z. It can also be seen that $b \parallel z$ implies that provided $b_3 \neq 0$, it holds that $\begin{pmatrix} y_1 & y_2 \end{pmatrix}^T = \begin{pmatrix} b_1/b_3 & b_2/b_3 \end{pmatrix}^T$, which is the *focus of expansion* mentioned as an unobservable case also for the observers presented in e.g. [11, 2, 9, 12].

7. Simulations

The performance of the estimators described in Sections 4 and 5 is illustrated using simulation examples.

A structure estimation example is first presented, using the estimator (31) applied to the dynamic system (28) as described in Section 4.

The observed object contains one feature point, executing a periodic motion used also in [3], and governed by the parameter vectors

$$\begin{aligned} \omega &= \begin{pmatrix} -0.4 & 0.5 & 4 \end{pmatrix}^T \\ b &= \begin{pmatrix} 0 & 2\pi\sin(2\pi t) & 2\pi\cos(2\pi t) \end{pmatrix}^T . \end{aligned} \tag{58}$$

A single feature point is observed, with an initial position selected as $x_0 = \begin{pmatrix} 0 & 2 & 3 \end{pmatrix}^T$.

The simulation is done using the initial values $\hat{\alpha}_0 = 1$, $\hat{z} = 0$, and we select $F = -10 \cdot \mathrm{I}$ and $Q = 750 \cdot \mathrm{I}$ as the matrices employed to determine the matrix P from (30).

A simulation result is shown in Figure 1. From Figure 1 it can be seen that the estimated parameter $\hat{\alpha}_0$ converges to its true value. This has the effect that also the 3D position estimates converge to their true values, as can be seen from the lower plot in Figure 1.

A structure and angular velocity estimation example is now presented, using the estimator (41) applied to the dynamic system (40) as described in Section 5.

The observed object contains one feature point, executing the same motion as in the previous example, shown in Figure 1. A single feature point is observed, with an initial position selected as $x_0 = \begin{pmatrix} 0 & 2 & 3 \end{pmatrix}^T$. The initial value for the estimated angular velocity vector was set to zero.

The simulation is done using the initial values $\hat{\alpha}_0 = 1$, $\hat{z} = 0$, and we select $F = -10 \cdot \mathrm{I}$ and

$$Q = 100 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

as the matrices employed to determine the matrix P from (30).

The simulation result is shown in Figure 2. From Figure 2 it can be seen that the estimated parameter $\hat{\alpha}_0$ converges to its true value, and that the angular velocity vector converges to its true value (which can be seen in (58)). This has the effect that also the 3D position estimates converge to their true values, as can be seen from the lower right plot in Figure 2.

8. Conclusions

Estimation of 3D structure and motion from 2D images can be achieved using a dynamic systems formulation, where nonlinear and adaptive observers can be used for estimation of states and parameters. In this paper we have demonstrated how a single parametrization of the underlying perspective dynamic system can be used for reformulation of the structure estimation problem as a parameter estimation problem, where a constant parameter is estimated. It is also shown how the so obtained structure estimator can be extended to also estimate the angular velocity.

A stability analysis is presented, showing convergence of the structure and angular velocity estimator using Lyapunov theory and persistency of excitation based arguments. The performance of the estimators for structure, and structure and angular velocity, is also demonstrated using simulation examples.

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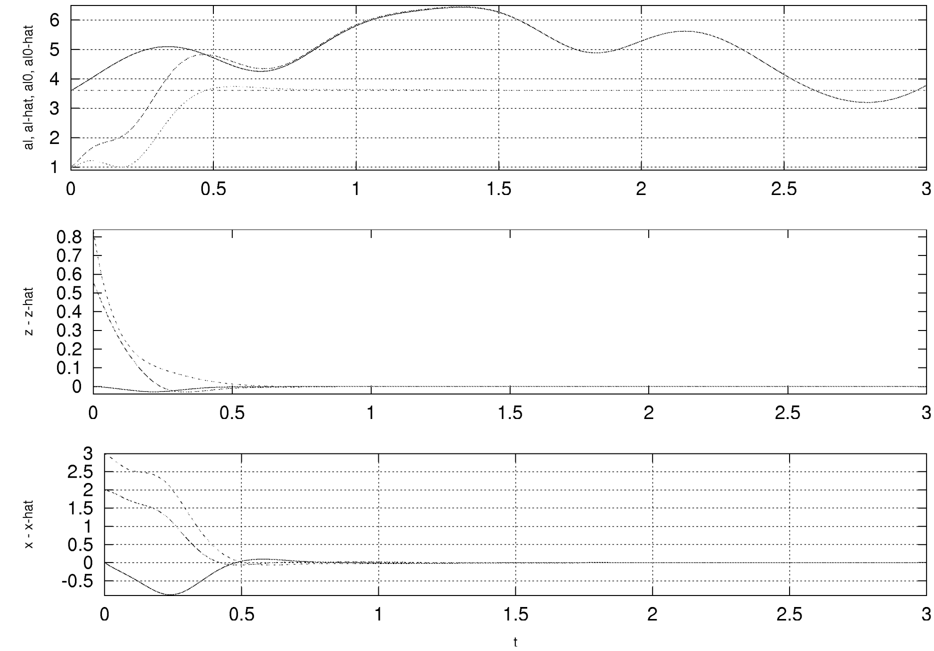

Figure 1. Structure estimation results, for a periodic object motion with the same angular and linear velocities as in [3]. True (solid) and estimated (dash-dotted) values of α_0 and α (top). Estimation errors $z - \hat{z}$ (middle) and $x - \hat{x}$ (bottom).

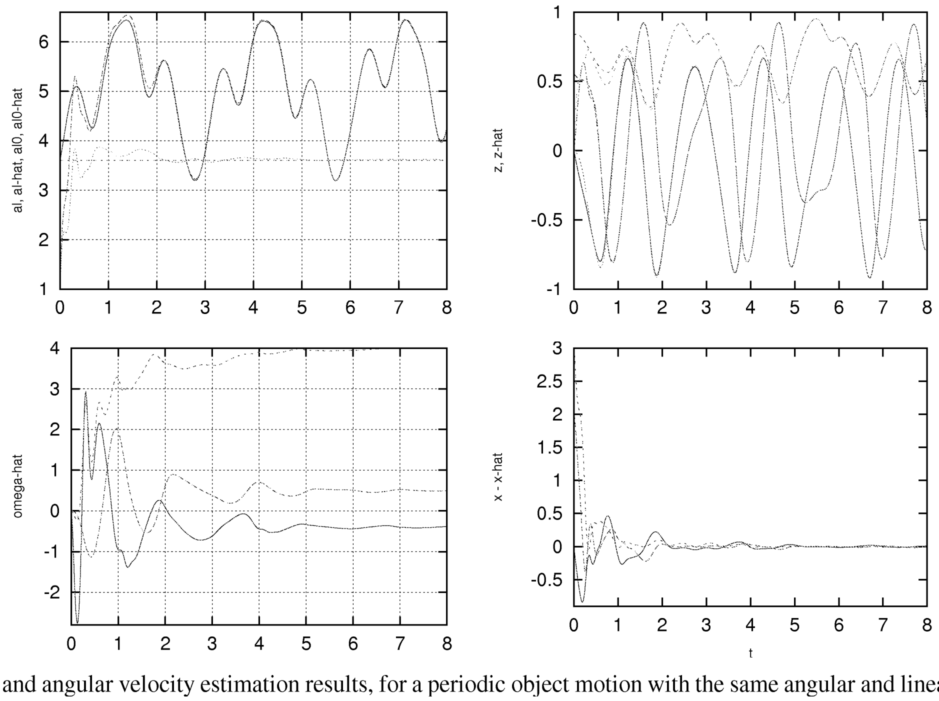

Figure 2. Structure and angular velocity estimation results, for a periodic object motion with the same angular and linear velocities as in [3]. True (solid) and estimated (dash-dotted) values of α_0 and α (upper left) and z (upper right). Estimated values of ω (lower left). Estimation errors for the 3D-position x (lower right). (The x-axis corresponds to time.)

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