Consensus Algorithms for Distributed Sensor Networks

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- "Consensus Algorithm" is an interaction rule that specifies the information exchange between an agent and all of its neighbors on the network.
- Tools for analysis of consensus: Matrix Theory, Algebraic Graph Theory, Control Theory

Applications

- Synchronization of Coupled Oscillators
  - Kuramoto model of coupled oscillators on a graph: $\dot{\theta}_i = k \sum_{j \in N_i} \sin(\theta_j - \theta_i) + \omega_i$
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- Rendezvous in Space
  - reaching a consensus in position by a number of agents with an interaction topology which is position induced.

Applications

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- Distributed Formation Control
  - moving in a formation is a cooperative task that requires consent and collaboration of every agent in the formation.
  - Local cost $U_i(x) = \sum_{j \in N_i} ||x_j - x_i - r_{ij}||^2$ where $x_i =$ position of vehicle i and $r_{ij} =$ desired inter-vehicle relative position vector.

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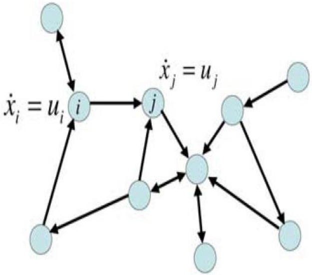
The two problems are cooperative tasks requiring the willing participation of all agents.

- Cooperation: giving consent to providing one's state and following a common protocol that serves the group objective.

Information Censensus



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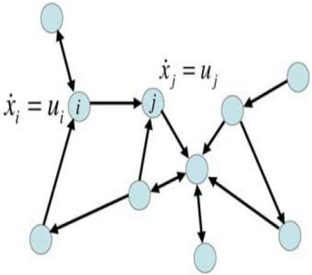

- A network of decision making agents with dynamics $\dot{x}_i = u_i$ on a graph $G = (V, E)$
- Goal is reaching consensus via local communication with neighbors, $x = \alpha 1$ with $1 = (1, \cdots, 1)^\top$, and $\alpha \in \mathbb{R}$ is the collective decision

Basics from Algebraic Graph Theory

- Graph denoted by $G = (V, E)$ with $V = \{1, 2, \cdots, n\}$, and $E = \{(i, j) \in V \times V : i \sim j\}$
- $A = [a_{ij}]$ is Adjacency Matrix
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  - useful for describing network topology of mobile sensor networks and flocks.
- Laplacian of graph: $L = D - A$ where $D = diag\{d_1, d_2, \cdots, d_n\}$ and $d_i = \sum_{j \neq i} a_{ij}$

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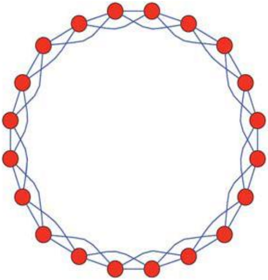
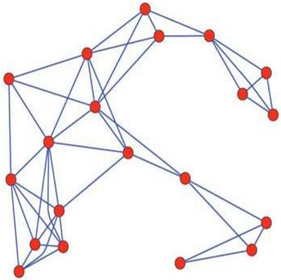
- If undirected graph G is connected, then $\lambda_2 > 0$.

Algebraic Connectivity



$\lambda_1 = 0, \lambda_2 = 0.48$

$\lambda_n = 6.24, \Delta = 4$



$\lambda_1 = 0, \lambda_2 = 0.25$

$\lambda_n = 9.37, \Delta = 8$

Information Consensus for undirected networks

- Distributed consensus algorithm:
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- Lemma: Let G be a connected undirected graph. Then the algorithm above asymptotically solves the "average consensus problem" for all initial states, $\alpha = 1/n \sum_{i=1}^{n} x_i(0)$.

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- Lemma (Spectral Localization): Let G be a strongly connected digraph on n nodes. Then $rank(L) = n - 1$ and all nontrivial eigenvalues of L have positive real parts. If G has $c \geq 1$ strongly connected components, then $rank(L) = n - c$.

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- Note: the above lemma holds under weaker condition of existence of a directed spanning tree. (there exists a node r such that all other nodes can be linked to r via a directed path.)
- Balanced Digraph: G is balanced if $\sum_{j \neq i} a_{ij} = \sum_{j \neq i} a_{ji} \Rightarrow 1^\top L = 0$. Thus, 1 is a left eigenvector of L with eigenvalue of 0.

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Lemma: Consider a network of n agents with topology G and the following consensus algorithm:
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- If the digraph is balanced, an average consensus is asymptotically reached.

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Lemma (Perron-Frobenius): Let P be a primitive matrix with left and right eigenvectors v, w so that $Pv = v$ and $w^\top P = w^\top$ with $v^\top w = 1$. Then, $\lim_{k\to\infty} P^k = vw^\top$.

Consensus in Discrete-Time

- Distributed consensus algorithm:
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- If G is strongly connected and $0 < \epsilon < 1/\Delta$, then P is a primitive matrix.

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Theorem: Consider a network of agents with topology G, and distributed consensus algorithm
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- The group decision value is $\alpha = \sum_i w_i x_i(0)$ with $\sum_i w_i = 1$.
- If the digraph is balanced, average consensus is asymptotically achieved, $\alpha = \sum_i x_i(0)/n$.

Performance Analysis of Consensus Algorithms

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- $\lambda_2 = min\ \delta^\top L\delta / \delta^\top \delta$ with $\lambda_2 = \lambda_2(L_s)$
- $\mu_2 = Max\ \delta^\top P\delta / \delta^\top \delta$ with $\mu_2 = 1 - \epsilon\lambda_2$

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Note: This results also holds for a strongly connected balanced digraph.

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- What if we want a weighted average consensus using the desired weighting vector $\gamma = (\gamma_1, \cdots, \gamma_n)$?
- Proposed dynamics: $K\dot{x} = -Lx$ with $K = diag(\gamma_1, \cdots, \gamma_n)$. Thus, each node updates its states by $\gamma_i \dot{x}_i = \sum_{j \in N_i} a_{ij}(x_j(t) - x_i(t))$.

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Theorem: The algorithm above, asymptotically solves the average consensus problem for a uniform one-hop time delay τ for all initial states, if $0 \le \tau < \pi/2\lambda_n$.

Consensus under communication time delays

- Suppose that agent i receives a message sent by its neighbor j after a time-delay of τ.
- For an undirected graph, consider the consensus algorithm: $\dot{x}_i = \sum_{j \in N_i} a_{ij}(x_j(t-\tau) - x_i(t-\tau)) \Rightarrow \dot{x} = -Lx(t-\tau)$.
- Laplace transform $\Rightarrow X(s) = H(s)/s\, x(0)$ where $H(s) = (I_n + e^{-s\tau} L)^{-1} \Rightarrow$ Nyquist Criterion for stability of $H(s)$

Theorem: The algorithm above, asymptotically solves the average consensus problem for a uniform one-hop time delay τ for all initial states, if $0 \leq \tau < \pi/2\lambda_n$.
Note: A sufficient condition for having average consensus under time delay is $\tau < \pi/4\Delta \Rightarrow$ trade-off between maximum degree of the network and robustness to time delays.

Alternative forms of consensus algorithms

- Consider $\dot{x}_i = 1/|N_i| \sum_{j \in N_i} (x_j(t) - x_i(t)) \Rightarrow \dot{x} = -Lx(t)$ where $L = I - D^{-1}A$

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  - Discrete representation: $x(k+1) = Px(k)$ where $P = I - \epsilon L$ with $0 < \epsilon < 1$. ($\Delta = 1$ here!)
  - If $\epsilon = 1$, does not converge for some digraphs such as cycles of length n.
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- For undirected networks, consider $x_i(k+1) = (x_i(k) + \sum_{j \in N_i} x_j(k))/(1 + |N_i|) \Rightarrow x(k+1) = (I+D)^{-1}(I+A)x(k)$

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Theorem: Consider a network of agents with consensus algorithm $\dot{x} = -L(G_k)x$ with $k = s(t) \in J$. Suppose every graph G_k is a balanced digraph which is strongly connected and let $\lambda_2^* = min\, \lambda_2(G_k)$. Then, for any arbitrary switching signal, the agents asymptotically reach an average consensus for all initial states with a speed faster than or equal to λ_2^*.

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- Let $P = \{P_1, \ldots, P_m\}$ denote the set of Perron matrices associated with a finite set of undirected graphs Γ with n self-loops. The switching network is "Periodically Connected" with $N > 1$ if the unions of all graphs over a sequence of intervals $[j, j + N)$ for j=0,1,... are connected graphs.

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- Lemma (Wolfowitz): Let $P = \{P_1, \ldots, P_m\}$ be a finite set of primitive stochastic matrices such that for any sequence of matrices $P_{s_k}, \ldots, P_{s_0} \in P$, with $k \geq 1$, the product $P_{s_k} \ldots P_{s_1} P_{s_0}$ is a primitive matrix. Then there exist a row vector $w^\top$ such that $\lim_{k\to\infty} P_{s_k} \ldots P_{s_1} P_{s_0} = 1w^\top$.

Consensus in Switching Networks

Theorem (Jadbabaie'03): Consider the system $x_{k+1} = P_{s_k} x_k$ with $P_{s_k} \in P$ for all k. Assume the switching network is periodically connected. Then, $\lim_{k \to \infty} x_k = \alpha 1$, meaning that an alignment is asymptotically reached.

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Note: w depends on the switching sequence and can not be determined a priori.