26. Suppose you are to design a unity feedback controller for a first-order plant depicted in Fig. 3.50. (As you will learn in Chapter 4, the configuration shown is referred to as a proportional-integral controller.) You are to design the controller so that the closed-loop poles lie within the shaded regions shown in Fig. 3.51.

![Unity feedback system for Problem 3.26](image)

Figure 3.50: Unity feedback system for Problem 3.26

![Desired closed-loop pole locations for Problem 3.26](image)

Figure 3.51: Desired closed-loop pole locations for Problem 3.26

(a) What values of $\omega_n$ and $\zeta$ correspond to the shaded regions in Fig. 3.51? (A simple estimate from the figure is sufficient.)

(b) Let $K_\alpha = \alpha = 2$. Find values for $K$ and $K_I$ so that the poles of the closed-loop system lie within the shaded regions.

(c) Prove that no matter what the values of $K_\alpha$ and $\alpha$ are, the controller provides enough flexibility to place the poles anywhere in the complex (left-half) plane.

**Solution:**

(a) The values could be worked out mathematically but working from the diagram:
\[ \sqrt{3^2 + 2^2} = 3.6 \implies 2.6 \leq \omega_n \leq 4.6 \]
\[ \theta = \sin^{-1} \zeta \]
\[ \zeta = \sin \theta. \]

From the figure:

\[ \theta \approx 34^\circ \quad \zeta_1 = 0.554 \]
\[ \theta \approx 70^\circ \quad \zeta_2 = 0.939 \]

\[ \implies 0.6 \leq \zeta \leq 0.9 \quad \text{(roughly)} \]

(b) Closed-loop pole positions:

\[
\begin{align*}
    s(s + \alpha) + (Ks + KK_I)K_n &= 0 \\
    s^2 + (\alpha + KK_n)s + KK_I K_n &= 0
\end{align*}
\]

For this case:

\[ s^2 + (2 + 2K)s + 2KK_I = 0 \quad (*) \]

Choose roots that lie in the center of the shaded region,

\[
\begin{align*}
    (s + (3 + j2))(s + (3 - j2)) &= s^2 + 6s + 13 = 0 \\
    s^2 + (2 + 2K)s + 2KK_I &= s^2 + 6s + 13 \\
    2 + 2K &= 6 \implies K = 2 \\
    13 &= 4K_I \implies K_I = \frac{13}{4}.
\end{align*}
\]

(c) For the closed-loop pole positions found in part (b), in the (*) equation the value of \( K \) can be chosen to make the coefficient of \( s \) take on any value. For this value of \( K \) a value of \( K_I \) can be chosen so that the quantity \( KK_I K_n \) takes on any value desired. This implies that the poles can be placed anywhere in the complex plane.
32. Consider the system shown in Fig. 3.55, where

\[ G(s) = \frac{1}{s(s + 3)} \quad \text{and} \quad D(s) = \frac{K(s + z)}{s + p}. \]  

Find \( K, z, \) and \( p \) so that the closed-loop system has a 10% overshoot to a step input and a settling time of 1.5 sec (1% criterion).

---

**Solution:**

For the 10% overshoot:

\[ M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 10\% \]

\[ \Rightarrow \zeta = \frac{\ln(M_p)^2}{\pi^2 + (\ln(M_p))^2} = 0. \]

For the 1.5sec (1% criterion):

\[ \omega_n = \frac{4.6}{\zeta T_s} = \frac{4.6}{(0.6)(1.5)} = 5.11. \]

The closed-loop transfer function is:

\[ \frac{Y(s)}{R(s)} = \frac{K \frac{s + z}{s + p} \times \frac{1}{s(s + 3)}}{1 + K \frac{s + z}{s + p} \times \frac{1}{s(s + 3)}} = \frac{K(s + z)}{s(s + 3)(s + p) + K(s + z)}. \]

**Method I.**

From inspection, if \( z = 3 \), \((s + 3)\) will cancel out and we will have a standard form transfer function. As perfect cancellation is impossible, assign \( z \) a value that is very close to 3, say 3.1. But in determining the \( K \) and \( p \), assume that \((s + 3)\) and \((s + 3.1)\) cancelled out each other. Then:

\[ \frac{Y(s)}{R(s)} = \frac{K}{s^2 + ps + K}. \]

As the additional pole and zero will degrade the system, pick some larger damping ration.
Let $\zeta = 0.7$

\[
\begin{align*}
\omega_n &= \frac{4.6}{\zeta t_s} = \frac{4.6}{(0.7)(1.5)} = 4.38, \text{ so let } \omega_n = 4.5 \\
p &= 2\zeta\omega_n = 2 \times 0.7 \times 4.5 = 6.3 \\
K &= \omega_n^2 = 20.25.
\end{align*}
\]

**Method II.**

There are 3 unknowns ($z$, $p$, $K$) and only 2 specified conditions. We can arbitrarily choose $p$ large such that complex poles will dominate in the system response.

Try $p = 10z$

Choose a damping ratio corresponding to an overshoot of 5% (instead of 10%, just to be safe).

$\zeta = 0.707$.

From the formula for settling time (with a 1% criterion)

\[
\omega_n = \frac{4.6}{\zeta t_s} = \frac{4.6}{0.707 \times 1.5} = 4.34
\]

adding some margin, let $\omega_n = 4.88$. The characteristic equation is

\[
Q(s) = s^3 + (s + p)s^2 + (3p + K)s + Kz = (s + a)(s^2 + 2\zeta\omega_n s + \omega_n^2)
\]

We want the characteristic equation to be the product of two factors, a couple of conjugated poles (dominant) and a non-dominant real pole far form the dominant poles.

Equate the coefficients of like powers of $s$ in the expressions of the characteristic equation.

\[
\begin{align*}
\omega_n^2 a &= Kz \\
2\zeta\omega_n a + \omega_n^2 &= 30z + K \\
2\zeta\omega_n + a &= 3 + 10z.
\end{align*}
\]

Solving the three equations we get

\[
\begin{align*}
z &= 5.77 \\
p &= 57.7 \\
K &= 222.45 \\
a &= 53.79.
\end{align*}
\]
35. Consider the following second-order system with an extra pole:

\[ H(s) = \frac{\omega_n^2 p}{(s + p)(s^2 + 2\zeta \omega_n s + \omega_n^2)} \]

Show that the unit step response is

\[ y(t) = 1 + Ae^{-pt} + Be^{-\sigma t} \sin(\omega_d t - \theta), \]

where

\[
A = \frac{-\omega_n^2}{\omega_n^2 - 2\zeta \omega_n p + p^2}, \\
B = \frac{p}{\sqrt{(p^2 - 2\zeta \omega_n p + \omega_n^2)(1 - \zeta^2)}}, \\
\theta = \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{-\zeta} + \tan^{-1} \frac{\omega_n \sqrt{1 - \zeta^2}}{p - \zeta \omega_n}.
\]

(a) Which term dominates \( y(t) \) as \( p \) gets large?
(b) Give approximate values for \( A \) and \( B \) for small values of \( p \).
(c) Which term dominates as \( p \) gets small? (Small with respect to what?)

(d) Using the preceding explicit expression for \( y(t) \) or the step command in MATLAB, and assuming that \( \omega_n = 1 \) and \( \zeta = 0.7 \), plot the step response of the preceding system for several values of \( p \) ranging from very small to very large. At what point does the extra pole cease to have much effect on the system response?

**Solution:**

Second-order system:

\[ H(s) = \frac{\omega_n^2 p}{(s + p)(s^2 + 2\zeta \omega_n s + \omega_n^2)} \]

Unit step response:

\[ Y(s) = \frac{1}{s} H(s), \quad y(t) = \mathcal{L}^{-1}\{Y(s)\} \]

\[ s^2 + 2\zeta \omega_n s + \omega_n^2 = (s + \sigma + j\omega_d)(s + \sigma - j\omega_d) \]

where

\[ \sigma = \zeta \omega_n, \quad \omega_d = \omega_n \sqrt{1 - \zeta^2}. \]

Thus from partial fraction expansion:

\[ Y(s) = \frac{k_1}{s} + \frac{k_2}{s + p} + \frac{k_3}{s + \sigma + j\omega_d} + \frac{k_4}{s + \sigma - j\omega_d} \]
solving for $k_1, k_2, k_3,$ and $k_4$:

$$k_1 = H(0) \implies k_1 = 1$$

$$k_2 = \frac{\omega_n^2 p}{s(s + \sigma + j\omega_d)(s + \sigma - j\omega_d)} \bigg|_{s = -p} \implies k_2 = \frac{-\omega_n^2}{\omega_n^2 - 2\rho \omega_n + p^2}$$

$$k_3 = (s + \sigma + j\omega_d)Y(s)|_{s = -\sigma - j\omega_d}$$

$$\implies k_3 = \frac{p}{2 \sqrt{(1 - \zeta^2)(p^2 - 2\rho \omega_n + \omega_n^2)}} e^{-i\theta} = |k_3| e^{-i\theta}$$

$$k_4 = k_3^*$$

where

$$\theta = \tan^{-1}\left(\frac{\sqrt{1 - \zeta^2}}{-\zeta}\right) + \tan^{-1}\left(\frac{\omega_n \sqrt{1 - \zeta^2}}{p - \zeta \omega_n}\right)$$

Thus

$$Y(s) = \frac{1}{s} + \frac{k_2}{s + p} + |k_3| \left(\frac{e^{-i\theta}}{s + \sigma + j\omega_d} + \frac{e^{+i\theta}}{s + \sigma - j\omega_d}\right)$$

Inverse Laplace:

$$y(t) = 1 + k_2 e^{-pt} + |k_3| \left(e^{-i\theta} e^{-(\sigma + j\omega_d)t} + e^{+i\theta} e^{-(\sigma - j\omega_d)t}\right)$$

or

$$y(t) = 1 + \frac{-\omega_n^2}{\omega_n^2 - 2\rho \omega_n + p^2} e^{-pt} + \frac{p}{\sqrt{(1 - \zeta^2)(p^2 - 2\rho \omega_n + \omega_n^2)}} e^{-\sigma t} \cos(\omega_d t + \theta)$$

(a) As $p$ gets large the $B$ term dominates.

(b) For small $p$: $A \approx -1, B \approx 0$.

(c) As $p$ gets small $A$ dominates.

(d) The effect of a change in $p$ is not noticeable above $p \approx 10$. 
Problem 3.35: Step responses.
38. Suppose that unity feedback is to be applied around the listed open-loop systems. Use Routh’s stability criterion to determine whether the resulting closed-loop systems will be stable.

(a) \( KG(s) = \frac{4(s+2)}{s(s^2+2s^2+3s+1)} \)

(b) \( KG(s) = \frac{2(s+4)}{s^3(s+1)} \)

(c) \( KG(s) = \frac{4(s^2+2s^2+s+1)}{s^3(s^2+2s^2-s-1)} \)

Solution:

(a) \[ 1 + KG = s^4 + 2s^3 + 3s^2 + 8s + 8 = 0 \]

\[
\begin{array}{c|ccc}
 s^4 & 1 & 3 & 8 \\
 s^3 & 2 & 8 \\
 s^2 & a & b \\
 s^1 & c \\
 s^0 & d \\
\end{array}
\]

where \( a = \frac{2 \times 3 - 8 \times 1}{2} = -1 \), \( b = \frac{2 \times 8 - 1 \times 0}{2} = 8 \)

\( c = \frac{3a - 2b}{a} = \frac{-8 - 16}{-1} = 24 \)

\( d = b = 8 \)

2 sign changes in first column \( \Rightarrow \) 2 roots not in LHP \( \Rightarrow \) unstable.

(b) \[ 1 + KG = s^3 + s^2 + 2s + 8 = 0 \]

The Routh’s array is,

\[
\begin{array}{c|cc}
 s^3 & 1 & 2 \\
 s^2 & 1 & 8 \\
 s^1 & -6 \\
 s^0 & 8 \\
\end{array}
\]

There are two sign changes in the first column of the Routh array. Therefore, there are two roots not in the LHP.
\[ 1 + KG = s^5 + 2s^4 + 3s^3 + 7s^2 + 4s + 4 = 0 \]

\[
\begin{align*}
    s^5 & : 1 & 3 & 4 \\
    s^4 & : 2 & 7 & 4 \\
    s^3 & : a_1 & a_2 \\
    s^2 & : b_1 & b_2 \\
    s^1 & : c_1 \\
    s^0 & : d_1
\end{align*}
\]

where

\[
\begin{align*}
    a_1 &= \frac{6 - 7}{2} = -\frac{1}{2} \\
    b_1 &= \frac{-7/2 - 4}{-1/2} = 15 \\
    c_1 &= \frac{30 + 2}{15} = \frac{32}{15} \\
    d_1 &= 4 \\
    a_2 &= \frac{8 - 4}{2} = 2 \\
    b_2 &= \frac{-4/2 - 0}{-1/2} = 4
\end{align*}
\]

2 sign changes in the first column \( \rightarrow \) 2 roots not in the LHP \( \rightarrow \) unstable.
42. Consider the system shown in Fig. 3.58.

(a) Compute the closed-loop characteristic equation.

(b) For what values of \((T, A)\) is the system stable? Hint: An approximate answer may be found using

\[ e^{-Ts} \approx 1 - Ts \]

or

\[ e^{-Ts} \approx \frac{1 - \frac{T}{2}s}{1 + \frac{T}{2}s} \]

for the pure delay. As an alternative, you could use the computer MATLAB (Simulink) to simulate the system or to find the roots of the system’s characteristic equation for various values of \(T\) and \(A\).

**Solution:**

(a) The characteristic equation is,

\[ s(s + 1) + Ae^{-Ts} = 0 \]

(b) Using \(e^{-Ts} \approx 1 - Ts\), the characteristic equation is,

\[ s^2 + (1 - TA)s + A = 0 \]

The Routh’s array is,

\[
\begin{array}{ccc}
  s^2 & 1 & A \\
  s^1 & 1 - TA & 0 \\
  s^0 & A & \\
\end{array}
\]

For stability we must have \(A > 0\) and \(TA < 1\).

Using \(e^{-Ts} \approx \frac{(1 - \frac{T}{2}s)}{(1 + \frac{T}{2}s)}\), the characteristic equation is,

\[ s^3 + \left(1 + \frac{2}{T}\right)s^2 + \left(\frac{2}{T} - A\right)s + \frac{2}{T}A = 0 \]

The Routh’s array is,
For stability we must have all the coefficients in the first column be positive. The following Simulink diagram simulates the system.

\[
\begin{align*}
    s^3 & : 1 \\
    s^2 & : \left(1 + \frac{2}{T}\right)
        \begin{pmatrix}
          \frac{2}{T} - A \\
          \frac{2A}{T}
        \end{pmatrix} \\
    s^1 & : \frac{(1 + \frac{2}{T}) \left(\frac{2}{T} - A\right) - \frac{2A}{T}}{(1 + \frac{2}{T})} \\
    s^0 & : \frac{2A}{T}
\end{align*}
\]

Problem 3.42: Simulink simulation diagram.