Exam 2: Unsupervised Learning (600.692)

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1. **Manifold Learning.** Let $\{\boldsymbol{x}_j \in \mathbb{R}^D\}_{j=1}^N$ be a set of points that lie approximately in a manifold of dimension d embedded in $\mathbb{R}^D$. Imagine you have applied KPCA with kernel κ and LLE with K-NN to the data. Assume now you are given a new point $\boldsymbol{x} \in \mathbb{R}^D$ and you wish to find its corresponding point $\boldsymbol{y} \in \mathbb{R}^d$ according to KPCA and LLE. How would you compute $\boldsymbol{y} \in \mathbb{R}^d$ without applying KPCA or LLE from scratch to the $N+1$ points? Under what conditions the solution you propose is equivalent to applying KPCA or LLE to the $N+1$ points?

2. **K-Subspaces.** Consider the objective function of the K-subspaces algorithm:

$$f(\{\boldsymbol{\mu}_i\}_{i=1}^n, \{U_i\}_{i=1}^n) = \min_{\substack{\{\boldsymbol{\mu}_i\}_{i=1}^n \\ \{U_i : U_i^\top U_i = I\}_{i=1}^n}} \sum_{j=1}^N \min_{i=1,\dots,n} \|(I - U_i U_i^\top)(\boldsymbol{x}_j - \boldsymbol{\mu}_i)\|^2. \tag{1}$$

   Let $\{\boldsymbol{\mu}_i^{(k)}\}_{i=1}^n, \{U_i^{(k)}\}_{i=1}^n$ be the estimates of the subspace parameters at the kth iteration of the K-subspaces algorithm. Show that the iterations of K-subspaces are such that

$$f(\{\boldsymbol{\mu}_i^{(k+1)}\}_{i=1}^n, \{U_i^{(k+1)}\}_{i=1}^n) \le f(\{\boldsymbol{\mu}_i^{(k)}\}_{i=1}^n, \{U_i^{(k)}\}_{i=1}^n). \tag{2}$$

3. **Low-Rank Subspace Clustering.** Let $X = [\boldsymbol{x}_1, \dots, \boldsymbol{x}_N] \in \mathbb{R}^{D\times N}$ be a data matrix whose columns are drawn from a union of n subspaces. Let $X = U\Sigma V^\top$ and $X = U_1\Sigma_1 V_1^\top$ be, respectively, the full and compact SVDs of X, with V partitioned as $[V_1, V_2]$, where $V_1 \in \mathbb{R}^{N\times r}$, $V_2 \in \mathbb{R}^{N\times(N-r)}$ and $\operatorname{rank}(X) = r$. Let us express each data point as a linear combination of all data points, i.e., for all j, $\boldsymbol{x}_j = \sum_{i=1}^N \boldsymbol{x}_i c_{ij}$, or equivalently $\boldsymbol{x}_j = X\boldsymbol{c}_j$, where $\boldsymbol{c}_j \in \mathbb{R}^N$. Let us now search for a matrix of coefficients $C = [\boldsymbol{c}_1, \dots, \boldsymbol{c}_N] \in \mathbb{R}^{N\times N}$ that solves the following optimization problem:

$$\min_C \ \|C\|_* + \frac{\lambda}{2}\|C\|_F^2 \ \text{ s.t. } \ X = XC \ \text{ and } \ C = C^\top, \tag{3}$$

   where $\lambda > 0$ is a parameter. Prove that $C^* = V_1 V_1^\top$.

   Hint: We showed in class that the solutions to $X = XC$ are of the form $C = V_1 V_1^\top + V_2 A$, for $A \in \mathbb{R}^{(N-r)\times N}$.