Two View Geometry

- From two views
  - Can recover motion: 8-point algorithm
  - Can recover structure: triangulation

- Why multiple views?
  - Get more robust estimate of structure: more data
  - No direct improvement for motion: more data & unknowns
**Why Multiple Views?**

- Cases that cannot be solved with two views
  - Uncalibrated case: need at least three views
  - Line features: need at least three views

- Some practical problems with using two views
  - Small baseline: good tracking, poor motion estimates
  - Wide baseline: good motion estimates, poor correspondences

- With multiple views one can
  - Track at high frame rate: tracking is easier
  - Estimate motion at low frame rate: throw away data

---

**Problem Formulation**

*Input:* Corresponding images (of "features") in multiple images.

*Output:* Camera motion, camera calibration, object structure.

\[
F(R_i, T_i, X^j) = \sum_{j=1}^{n} \sum_{i=1}^{m} \left\| \pi(R_iX^j + T_i) - x_i^j \right\|^2
\]

unknowns

---

Course #23: Multiple-View Geometry for Image-Based Modeling. Lecture F: Geometry and Algebra of Multiple Views.
Multiframe SFM as an Optimization Problem

- Can we minimize the re-projection error?

\[ F(R_i, T_i, X^j) = \sum_{j=1}^{n} \sum_{i=1}^{m} \left\| x_i^j - \pi(R_iX^j + T_i) \right\|^2 \]

- Number of unknowns = 3n + 6 (m-1) - 1
- Number of equations = 2nm

- Very likely to converge to a local minima
- Need to have a good initial estimate

Motivating Examples

Image courtesy of Paul Debevec

Course #23: Multiple-View Geometry for Image-Based Modeling, Lecture F: Geometry and Algebra of Multiple Views
Motivating Examples

Image courtesy of Kun Huang

Using Geometry to Tackle the Optimization Problem

- What are the basic relations among multiple images of a point/line?
  - Geometry and algebra

- How can I use all the images to reconstruct camera pose and scene structure?
  - Algorithm

- Examples
  - Synthetic data
  - Vision based landing of unmanned aerial vehicles
**Projection: Point Features**

Homogeneous coordinates of a 3-D point $p$
\[ X = [X, Y, Z, W]^T \in \mathbb{R}^4, \quad (W = 1) \]

Homogeneous coordinates of its 2-D image
\[ x = [x, y, z]^T \in \mathbb{R}^3, \quad (z = 1) \]

Projection of a 3-D point to an image plane
\[ \lambda(t)x(t) = \Pi(t)X \]
\[ \lambda(t) \in \mathbb{R}, \quad \Pi(t) = \begin{bmatrix} R(t) & T(t) \end{bmatrix} \in \mathbb{R}^{3 \times 4} \]
\[ R(t) \rightarrow A(t)R(t), \quad T(t) \rightarrow A(t)T(t) \]

**Multiple View Matrix for Point Features**

- **WLOG** choose frame 1 as reference
\[ x_i \times (\lambda_i x_i = \lambda_i R_i x_1 + T_i) \leftrightarrow \begin{bmatrix} \bar{x}_1 & x_1 \\ \bar{x}_2 & x_2 \\ \bar{x}_3 & x_3 \\ \vdots & \vdots \\ \bar{x}_m & x_m \end{bmatrix} \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} = 0 \]

\[ M_p = \begin{bmatrix} \bar{x}_2 R_2 x_1 \\ \bar{x}_2 T_2 \\ \bar{x}_3 R_3 x_1 \\ \bar{x}_3 T_3 \\ \vdots \\ \bar{x}_m R_m x_1 \\ \bar{x}_m T_m \end{bmatrix} \in \mathbb{R}^{3(m-1) \times 2} \]

- Rank deficiency of Multiple View Matrix
\[ \text{rank}(M_p) \leq 1 \]
\[ \text{rank}(M_p) = 1 \text{ (generic)} \]
\[ \text{rank}(M_p) = 0 \text{ (degenerate)} \]
Geometric Interpretation of Multiple View Matrix

\[ M_p = \begin{bmatrix} \tilde{x}_2 R_2 \tilde{x}_1 & \tilde{x}_2 T_2 \\ \tilde{x}_3 R_3 \tilde{x}_1 & \tilde{x}_3 T_3 \\ \vdots & \vdots \\ \tilde{x}_m R_m \tilde{x}_1 & \tilde{x}_m T_m \end{bmatrix} \in \mathbb{R}^{3(m-1) \times 2} \]

- Entries of \( M_p \) are normals to epipolar planes
- Rank constraint says normals must be parallel

Multiple View Matrix for Point Features

\[ M_p = \begin{bmatrix} \tilde{x}_2 R_2 \tilde{x}_1 & \tilde{x}_2 T_2 \\ \tilde{x}_3 R_3 \tilde{x}_1 & \tilde{x}_3 T_3 \\ \vdots & \vdots \\ \tilde{x}_m R_m \tilde{x}_1 & \tilde{x}_m T_m \end{bmatrix} \in \mathbb{R}^{3(m-1) \times 2} \]

\[ 0 \leq \text{rank}(M_p) \leq 1 \]

\( M_p \) encodes exactly the 3-D information missing in one image.

\[ \text{rank}(M_p) = 1 \quad \text{rank}(M_p) = 0 \]

Course: #23 Multiple-View Geometry for Image-Based Modeling, Lecture F: Geometry and Algebra of Multiple Views
Rank Conditions vs. Multifocal Tensors

\[
\begin{bmatrix}
\tilde{x}_2 R_2 x_1 & \tilde{x}_2 T_2 \\
\tilde{x}_3 R_3 x_1 & \tilde{x}_3 T_3 \\
\tilde{x}_m R_m x_1 & \tilde{x}_m T_m
\end{bmatrix}
\]

\[\text{rank} = 1 \Rightarrow x_i^T R_i x_1 = 0\]

Two views: epipolar constraint

\[
\begin{bmatrix}
\tilde{x}_2 R_2 x_1 & \tilde{x}_2 T_2 \\
\tilde{x}_3 R_3 x_1 & \tilde{x}_3 T_3 \\
\tilde{x}_m R_m x_1 & \tilde{x}_m T_m
\end{bmatrix}
\]

\[\text{rank} = 1 \Rightarrow x_i^T(R_i x_1 T_i^T - T_i x_1^T R_i^T) x_j = 0\]

Three views: trilinear constraints

- Other relationships among four or more views, e.g. quadrilinear constraints, are algebraically dependent!

Source: #23: Multiple-View Geometry for Image-Based Modeling, Lecture F: Geometry and Algebra of Multiple Views

Reconstruction Algorithm for Point Features

- Given m images of n points \((x_1^i, \ldots, x_m^i))_{i=1, \ldots, m}

\[
\lambda^i \begin{bmatrix}
\tilde{x}_2 R_2 x_1 & \tilde{x}_2 T_2 \\
\tilde{x}_3 R_3 x_1 & \tilde{x}_3 T_3 \\
\tilde{x}_m R_m x_1 & \tilde{x}_m T_m
\end{bmatrix} = 0 \in \mathbb{R}^{3(m-1) \times 1}
\]

\[
P_i \left[ \begin{bmatrix} R_i^T \\ T_i \end{bmatrix} \right] = \left[ \begin{array}{c}
\lambda^1 x_1^i \otimes \tilde{x}_2^i \\
\lambda^2 x_1^i \otimes \tilde{x}_3^i \\
\lambda^n x_1^i \otimes \tilde{x}_m^i
\end{array} \right] \left[ \begin{bmatrix} R_i^T \\ T_i \end{bmatrix} \right] = 0 \in \mathbb{R}^{3n \times 1}
\]

If \(n \geq 6\), in general \(\text{rank}(P_i) = 11\)

Source: #23: Multiple-View Geometry for Image-Based Modeling, Lecture F: Geometry and Algebra of Multiple Views
Reconstruction Algorithm for Point Features

Given $m$ images of $n(>6)$ points

For the $j^{th}$ point

$$\lambda^j \begin{bmatrix} \tilde{x}_2^j R_2 x_1^j \\ \tilde{x}_3^j R_3 x_1^j \\ \tilde{x}_m R_m x_1^j \end{bmatrix} + \begin{bmatrix} \tilde{x}_2^j T_2 \\ \tilde{x}_3^j T_3 \\ \tilde{x}_m T_m \end{bmatrix} = 0$$

SVD

$$\begin{bmatrix} \lambda^j \\ 1 \end{bmatrix} = 0 \Rightarrow \lambda^j$$

Iteration

For the $i^{th}$ image

$$\begin{bmatrix} \lambda^1 x_1^i \otimes x_2^i \otimes x_3^i \\ \lambda^2 x_1^i \otimes x_2^i \otimes x_3^i \\ \lambda^3 x_1^i \otimes x_2^i \otimes x_3^i \end{bmatrix} \begin{bmatrix} R_i^* \\ T_i^* \end{bmatrix} = 0 \Rightarrow (R_i^*, T_i^*)$$

If $n \geq 6$, in general rank$(P_i) = 11$

Reconstruction Algorithm for Point Features

1. Initialization
   - Set $k=0$
   - Compute $(R_2, T_2)$ using the 8-point algorithm
   - Compute $\lambda_1^j = \lambda_k^j$ and normalize so that $\alpha_1^j = 1$

2. Compute $(\tilde{R}_i, \tilde{T}_i)$ as the null space of $P_i=2, ..., m$.

3. Compute new $\lambda_j^k = \lambda_j^{k+1}$ as the null space of $M_j^k, j = 1, \ldots, n$. Normalize so that $\lambda_n^{k+1} = 1$

4. If $\|\lambda_k - \lambda_{k+1}\| \leq \epsilon$ stop, else $k = k+1$ and goto 2.

Course #25: Multiple-View Geometry for Image-Based Modeling. Lecture 5: Geometry and Algebra of Multiple Views.
Reconstruction Algorithm for Point Features

Course #23: Multiple-View Geometry for Image-Based Modeling, Lecture 7: Geometry and Algebra of Multiple Views
Multiple View Matrix for Line Features

Homogeneous representation of a 3-D line \( \underline{L} \)
\[
X = X_o + \mu \underline{V}, \quad X_o, \underline{V} \in \mathbb{R}^4, \mu \in \mathbb{R}
\]

Homogeneous representation of its 2-D image
\[
\underline{t} = [a, b, c]^T \in \mathbb{R}^3 \quad ax + by + c = 0
\]

Projection of a 3-D line to an image plane
\[

U(t)^T \pi(x(t)) = U(t)^T \pi(t)X = 0
\]

\[
\pi(t) = [R(t), T(t)] \in \mathbb{R}^{3 \times 4}
\]

Multiple View Matrix for Line Features

- **Point Features**

\[
M_p = \begin{bmatrix}
\tilde{x}_2 R x_1 & \tilde{x}_3 T_2 \\
\tilde{x}_3 R x_1 & \tilde{x}_3 T_3 \\
\tilde{x}_m R_m x_1 & \tilde{x}_m T_m
\end{bmatrix} \in \mathbb{R}^{2(m-1) \times 2}, \quad M_l = \begin{bmatrix}
\tilde{L}_2 R L_1 & \tilde{L}_3 T_2 \\
\tilde{L}_3 R L_1 & \tilde{L}_3 T_3 \\
L_m R_m L_1 & L_m T_m
\end{bmatrix} \in \mathbb{R}^{(m-1) \times 4}
\]

\[
\text{rank}(M) = 1
\]

- **Line Features**

\[
M_p \left( \begin{array}{c}
\tilde{\lambda} \\
1
\end{array} \right) = 0
\]

\[
M_l \begin{bmatrix}
\begin{bmatrix}
\begin{bmatrix}

M encodes exactly the 3-D information missing in one image.

Course #23: Multiple-View Geometry for Image-Based Modeling. Lecture 1: Geometry and Algebra of Multiple Views
Multiple View Matrix for Line Features

\[ M_l = \begin{bmatrix} l_1^T R_2 l_1 & l_1^T R_3 l_1 & l_1^T R_4 l_1 & l_1^T R_5 l_1 \\ l_2^T R_2 l_2 & l_2^T R_3 l_2 & l_2^T R_4 l_2 & l_2^T R_5 l_2 \\ l_3^T R_2 l_3 & l_3^T R_3 l_3 & l_3^T R_4 l_3 & l_3^T R_5 l_3 \\ l_4^T R_2 l_4 & l_4^T R_3 l_4 & l_4^T R_4 l_4 & l_4^T R_5 l_4 \end{bmatrix} \in \mathbb{R}^{4 \times 4}, \text{ rank}(M_l) = 3, 2, 1. \\
\text{each is an image of a (different) line in 3-D:}

- Rank = 3 any lines
- Rank = 2 intersecting lines
- Rank = 1 same line

Course #03: Multiple-View Geometry for Image-Based Modeling. Lecture #0: Geometry and Algebra of Multiple Views

Multiple View Matrix for Line Features

- \( \text{rank}(M_p) = 1 \) \\
\[ x_i (T_i x_i^T R_j - R_j x_i^T T_i) x_j = 0 \]

- \( \text{rank}(M_l) = 1 \) \\
\[ l_j^T T_j l_i^T R_j l_i - l_i^T T_i l_j^T R_j l_j = 0 \]

- \( \text{rank}(M_p) = 0 \) \\

- \( \text{rank}(M_l) = 0 \)
Reconstruction Algorithm for Line Features

1. Initialization
   - Set \( k=0 \)
   - Compute \((R_2, T_2)\) and \((R_3, T_3)\) using linear algorithm
   - Compute \( \lambda^j = \chi^j_k \) and normalize so that \( \alpha^j_k = 1 \)

2. Compute \( (\tilde{R}_i, \tilde{T}_i) \) as the null space of \( P_i = 2, \ldots, m \).

3. Compute new \( \lambda^j = \chi^j_{k+1} \) as the null space of \( M^j, j = 1, \ldots, n \). Normalize so that \( \lambda^j_{k+1} = 1 \)

4. If \( \| \lambda_k - \lambda_{k+1} \| \leq \epsilon \) stop, else \( k = k+1 \) and goto 2.

Source #29: Multiple-View Geometry for Image-Based Modeling, Lecture 5: Geometry and Algebra of Multiple Views

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Reconstruction Algorithm for Line Features

1. Initialization
   - Use linear algorithm from 3 views to obtain initial estimate for \((R_2, T_2)\) and \((R_3, T_3)\)

2. Given motion parameters compute the structure (equation of each line in 3-D)

3. Given structure compute motion

4. Stop if error is small stop, else goto 2.

Source #29: Multiple-View Geometry for Image-Based Modeling, Lecture 5: Geometry and Algebra of Multiple Views
Universal Rank Constraint

- What if I have both point and line features?
  - Traditionally points and lines are treated separately
  - Therefore, joint incidence relations not exploited

- Can we express joint incidence relations for
  - Points passing through lines?
  - Families of intersecting lines?

Universal Rank Constraint

- **The Universal Rank Condition** for images of a point on a line

\[
M = \begin{bmatrix}
D_1^1 & D_1^2 & D_2^1 & D_2^2 T_2 \\
D_3^1 & D_3^2 & D_3^3 & D_3^2 T_3 \\
D_m^1 & D_m^2 & D_m^3 & D_m^1 T_m
\end{bmatrix}
\]

where

- \( D_i \) = \( x_i \) or \( \bar{x}_i \) or \( \bar{t}_i \)

1. If \( D_1 = \bar{I}_1 \) and \( D_i^\perp = \bar{E}_i \) for some \( i \geq 2 \), then:

\[
1 \leq \text{rank}(M) \leq 2.
\]

- **Multi-nonlinear constraints** among 3, 4-wise images.

2. Otherwise:

\[
0 \leq \text{rank}(M) \leq 1.
\]

- **Multi-linear constraints** among 2, 3-wise images.
Universal Rank Constraint: points and lines

\[ M = \begin{bmatrix}
\tilde{x}_2R\tilde{a}_1 & \tilde{x}_2T_2 \\
\tilde{x}_3R\tilde{a}_1 & \tilde{x}_3T_3 \\
\tilde{x}_4R\tilde{a}_1 & \tilde{x}_4T_4 \\
\tilde{l}_5R\tilde{a}_1 & \tilde{l}_5T_5
\end{bmatrix} \in \mathbb{R}^{10 \times 4}, \quad 1 \leq \text{rank}(M) \leq 2. \]

\[ \text{rank}(M) = 2 \]

\[ \text{rank}(M) = 1 \]

Universal Rank Constraint: family of intersecting lines

\[ \tilde{M}_l = \begin{bmatrix}
\tilde{l}_2R\tilde{a}_1 & \tilde{l}_2T_2 \\
\tilde{l}_3R\tilde{a}_1 & \tilde{l}_3T_3 \\
\tilde{l}_4R\tilde{a}_1 & \tilde{l}_4T_4 \\
\tilde{l}_5R\tilde{a}_1 & \tilde{l}_5T_5
\end{bmatrix} \in \mathbb{R}^{8 \times 4}, \quad 1 \leq \text{rank}(\tilde{M}_l) \leq 2. \]

\[ l_1, l_2, l_3, l_4, l_5 \text{ each can randomly take the image of any of the lines:} \]

\[ \text{Nonlinear constraints among up to four views} \]
Universal Rank Constraint: multiple images of a cube

Three edges intersect at each vertex.

\[ M^j = \begin{bmatrix}
    x^j_1 R_0 x^1_1 & x^j_2 T_2 \\
    l_2^T R_0 x^1_1 & l_2^T T_2 \\
    l_3^T R_0 x^1_1 & l_3^T T_2 \\
    x^m_1 R_m x^m_1 & x^m_2 T_m \\
    l_2^T R_m x^m_1 & l_2^T T_m \\
    l_3^T R_m x^m_1 & l_3^T T_m \\
    1 & 0
\end{bmatrix} \]

0 ≤ rank(\(M^j\)) ≤ 1

Course #23: Multiple-View Geometry for Image-Based Modeling, Lecture F: Geometry and Algebra of Multiple Views
Universal Rank Constraint: multiple images of a cube

Example Graph 1:

Example Graph 2:

Course #29: Multiple-View Geometry for Image-Based Modeling. Lecture F: Geometry and Algebra of Multiple Views.
Multiple View Matrix for Coplanar Point Features

Homogeneous representation of a 3-D plane $\pi$

$$aX + bY + cZ + d = 0.$$  

$$\pi X = 0, \quad \pi = [\pi^1, \pi^2] : \pi^1 \in \mathbb{R}^3, \pi^2 \in \mathbb{R}$$

$$M = \begin{bmatrix}
D_{12}R_2D_1 & D_{12}T_2 \\
D_{23}R_3D_1 & D_{23}T_3 \\
\vdots & \vdots \\
D_{m1}R_mD_1 & D_{m1}T_m \\
\pi^1D_1 & \pi^2
\end{bmatrix}$$

Corollary [Coplanar Features]

Rank conditions on the new extended $M$ remain exactly the same!

Course #25: Multiple-View Geometry for Image-Based Modeling, Lecture F: Geometry and Algebra of Multiple Views

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Multiple View Matrix for Coplanar Point Features

Given that a point and line features lie on a plane $\pi$ in 3-D space:

In addition to previous constraints, it simultaneously gives homography:

$$\bar{e}_1(R_1\pi^2 - T_1\pi^1)x_1 = 0$$

$$0 \leq \text{rank}(M_p) \leq 1$$

$$\bar{I}^T(R_1\pi^2 - T_1\pi^1)\bar{I} = 0$$

$$0 \leq \text{rank}(M_t) \leq 1$$

$$M_p = \begin{bmatrix}
\bar{e}_2R_2e_1 & \bar{e}_2T_2 \\
\bar{e}_3R_3e_1 & \bar{e}_3T_3 \\
\bar{e}_mR_me_1 & \bar{e}_mT_m \\
\pi^1e_1 & \pi^2
\end{bmatrix} \in \mathbb{R}^{(3m-2) \times 2}, \quad M_t = \begin{bmatrix}
\bar{I}^T\bar{R} & \bar{I}^T\bar{T} \\
\bar{I}^T R_1 & \bar{I}^T T_1 \\
\bar{I}^T R_2 & \bar{I}^T T_2 \\
\bar{I}^T R_3 & \bar{I}^T T_3 \\
\bar{I}^T R_m & \bar{I}^T T_m \\
\pi M_1 & \pi^2
\end{bmatrix} \in \mathbb{R}^{m \times 4}$$

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Example: Vision-based Landing of a Helicopter

Landing on the ground

Tracking two meter waves

Vision-Based Landing of a UAV

Omid Shakernia
Dept of EECS, UC Berkeley
http://robotics.eecs.berkeley.edu/~omidsh

Course #23: Multiple-View Geometry for Image-Based Modeling, Lecture 5: Geometry and Algebra of Multiple Views
Example: Vision-based Landing of a Helicopter

Comparison of Motion Estimation Algorithms

General Rank Constraint for Dynamic Scenes

For a fixed camera, assume the point moves with constant acceleration:

\[ X(t) = X_0 + v_0 t + \frac{1}{2} a t^2 \]

\[ \bar{X} = \begin{bmatrix} X_0 \\ v_0 \\ a \end{bmatrix} \in \mathbb{R}^{10} \]

\[ \Pi(t) = \begin{bmatrix} I & t & t^2/2 & 0 \end{bmatrix} \in \mathbb{R}^{3 \times 10} \]

\[ \lambda(t) x(t) = \Pi(t) \bar{X}, \quad X \in \mathbb{R}^{10}, x \in \mathbb{R}^{3} \]

Before: \[ \lambda(t) x(t) = \Pi(t) X, \quad X \in \mathbb{R}^{4}, x \in \mathbb{R}^{3}. \]

Now: \[ \lambda(t) x(t) = \Pi(t) X(t), \quad X(t) = [b_1(t), b_2(t), \ldots, b_{n+1}(t)] \bar{X} \]

\[ \lambda(t) x(t) = \Pi(t) \bar{X}, \quad \bar{X} \in \mathbb{R}^{n+1}, x \in \mathbb{R}^{3}. \]

Time base
General Rank Constraint for Dynamic Scenes

Projection from $n$-dimensional space to $k$-dimensional space:

$$\lambda_i w_i = \bar{\Pi}_i x_i, \quad i = 1, \ldots, m,$$

$$\bar{\Pi}_i = \begin{bmatrix} \bar{R}_i & \bar{T}_i \end{bmatrix},$$

$$\bar{R}_i \in \mathbb{R}^{(k+1) \times (k+1)}, \quad \bar{T}_i \in \mathbb{R}^{(k+1) \times (n-k)}$$

We define the multiple view matrix as:

$$M = M = \begin{bmatrix} (D_{r1}^T R_2 D_1 & (D_{r2}^T T_2 \\ (D_{r3}^T R_3 D_1 & (D_{r3}^T T_3 \\ \vdots & \vdots \\ (D_{rm}^T R_m D_1 & (D_{rm}^T T_m \end{bmatrix}$$

where $D_i$'s and $D_i^\perp$'s are images and coinages of hyperplanes.
General Rank Constraint for Dynamic Scenes

Projection from $\mathbb{R}^n$ to $\mathbb{R}^k$

$D_1 = s_1 \ D_1^\perp = s_1^\perp$

$0 \leq \text{rank}(M) \leq (n - k)$

Projection from $\mathbb{R}^3$ to $\mathbb{R}^2$

$D_1 = s_1 \ D_1^\perp = s_1^\perp$

$0 \leq \text{rank}(M) \leq 1$

Summary

- Incidence relations $\leftrightarrow$ rank conditions
- Rank conditions $\Rightarrow$ multiple-view factorization
- Rank conditions imply all multi-focal constraints
- Rank conditions for points, lines, planes, and (symmetric) structures.
Incidence Relations among Features

"Pre-images" are all incident at the corresponding features

Traditional Multifocal or Multilinear Constraints

- Given \( m \) corresponding images of \( n \) points \( x_i^j \)

\[
\lambda_i^j x_i^j = \pi_i^j X^j \quad \pi_i^j = [R_i, T_i]
\]

- This set of equations is equivalent to

H. Heyden et al.

\[
\begin{bmatrix}
\Pi_1 & x_1 & 0 & \cdots & 0 \\
\Pi_2 & 0 & x_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\Pi_m & 0 & \cdots & 0 & x_m
\end{bmatrix}
\begin{bmatrix}
X \\
-\lambda_1 \\
-\lambda_2 \\
-\lambda_m
\end{bmatrix}
= 0
\]

O. Faugeras et al.

\[
\begin{bmatrix}
x_1 \Pi_1 \\
x_2 \Pi_2 \\
x_m \Pi_m
\end{bmatrix}
\begin{bmatrix}
X \\
0 \\
0
\end{bmatrix}
= 0
\]

- Multilinear constraints among 2, 3, 4 views

\[
H_p \in \mathbb{R}^{3m \times (m+4)} \\
\det(H_{(m+4) \times (m+4)}) = 0
\]

\[
F_p \in \mathbb{R}^{3m \times 4} \\
\det(F_{4 \times 4}) = 0
\]
**Multiview Formulation:**
The Multiple View Matrix

\[
M \triangleq \begin{bmatrix}
D_1^2 R_2 D_1 & D_1^2 T_2 \\
D_1^2 R_3 D_1 & D_1^2 T_3 \\
D_1^2 R_m D_1 & D_1^2 T_m
\end{bmatrix}, \quad \text{where} \quad \begin{cases}
D_i \doteq x_i \text{ or } \bar{t}_i \\
D_i^\perp \doteq \bar{x}_i \text{ or } \bar{t}_i^\perp
\end{cases}
\]

**Point Features**

\[
M_p = \begin{bmatrix}
\tilde{x}_2 R_2 x_1 & \tilde{x}_2 T_2 \\
\tilde{x}_3 R_3 x_1 & \tilde{x}_3 T_3 \\
\bar{x}_m R_m x_1 & \bar{x}_m T_m
\end{bmatrix} \in \mathbb{R}^{3(m-1) \times 2}, \quad M_l = \begin{bmatrix}
\tilde{t}_2 R_2 \bar{t}_1 & \tilde{t}_2 T_2 \\
\tilde{t}_3 R_3 \bar{t}_1 & \tilde{t}_3 T_3 \\
\bar{t}_m R_m \bar{t}_1 & \bar{t}_m T_m
\end{bmatrix} \in \mathbb{R}^{3(m-1) \times 4}
\]

\[0 \leq \text{rank}(M_p) \leq 1 \quad 0 \leq \text{rank}(M_l) \leq 1\]

*Course #23: Multiple-View Geometry for Image-Based Modeling, Lecture F: Geometry and Algebra of Multiple Views*