3.A Basic photometry with light sources and surfaces

In this section we give a concise description of a basic radiometric image formation model, and show that some simplifications are necessary in order to reduce the model to a purely geometric one, as described in this chapter. The idea is to describe how the intensity at a pixel on the image is generated. Under suitable assumptions, we show that such intensity depends only on the amount of energy radiated from visible surfaces in space and not on the vantage point.

Let S be a smooth visible surface in space; we denote the tangent plane to the surface at a point p by T_pS and its outward unit normal vector by ν_p. At each point $p \in S$ we can construct a local coordinate frame with its origin at p, its z-axis parallel to the normal vector ν_p, and its xy-plane parallel to T_pS (see Figure 3.13). Let L be a smooth surface that is irradiating light, which we call the *light source*. For simplicity, we may assume that L is the only source of light in space. At a point $q \in L$, we denote with T_qS and ν_q the tangent plane and the outward unit normal of L, respectively, as shown in Figure 3.13.

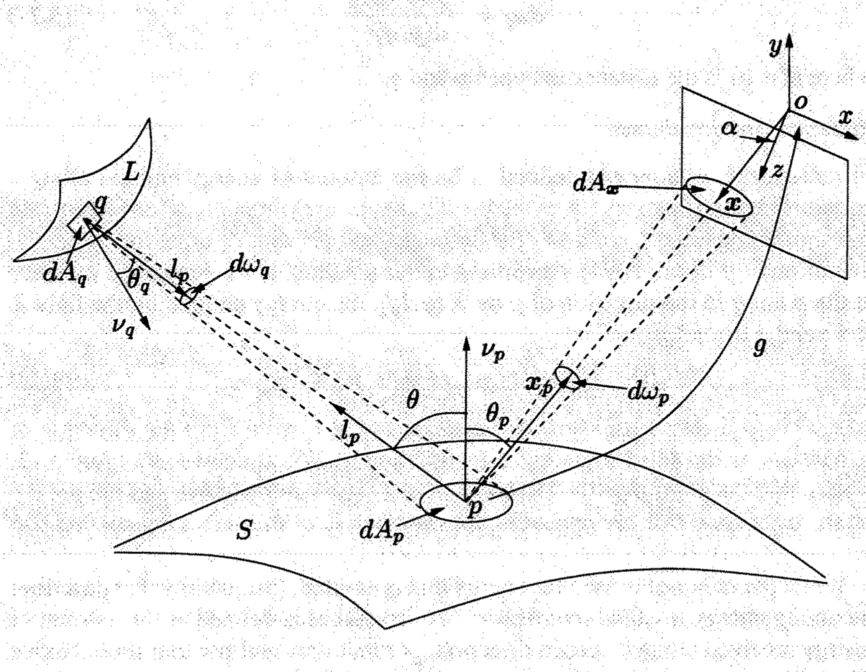

Figure 3.13. Generative model.

The change of coordinates between the local coordinate frame at p and the camera frame, which we assume coincides with the world frame, is indicated by a rigid-body transformation g; then g maps coordinates in the local coordinate frame at p into those in the camera frame, and any vector u in the local coordinate frame to a vector $v = g_*(u)$ in the camera frame.[8]

Foreshortening and solid angle

When considering interactions between a light source and a surface, we need to introduce the notion of *foreshortening* and that of *solid angle*. Foreshortening encodes how the light distribution on a surface changes as we change the surface orientation with respect to the source of illumination. In formulas, if dA_p is the area element in T_pS, and l_p is the unit vector that indicates the direction from p to q (see Figure 3.13), then the corresponding foreshortened area as seen from q is

$$\cos(\theta)dA_p,$$

where θ is the angle between the direction l_p and the normal vector ν_p; i.e. $\cos(\theta) = \langle \nu_p, l_p \rangle$. A solid angle is defined to be the area of a cone cut out on a unit sphere. Then, the infinitesimal solid angle $d\omega_q$ seen from a point q of the infinitesimal area dA_p is

$$d\omega_q \doteq \frac{\cos(\theta)dA_p}{d(p,q)^2}, \tag{3.27}$$

where $d(p, q)$ is the distance between p and q.

Radiance and irradiance

In radiometry, *radiance* is defined to be the amount of energy emitted along a certain direction, per unit area perpendicular to the direction of emission (the foreshortening effect), per unit of solid angle, and per unit of time, following the definition in [Sillion, 1994]. According to our notation, if we denote the radiance at the point q in the direction of p by $\mathcal{R}(q, l_p)$, the energy emitted by the light L at a point q toward p on S is

$$dE(p, l_p) \doteq \mathcal{R}(q, l_p) \cos(\theta_q)\, dA_q\, d\omega_q\, dt, \tag{3.28}$$

where $\cos(\theta_q)\, dA_q$ is the foreshortened area of dA_q seen from the direction of p, and $d\omega_q$ is the solid angle given in equation (3.27), as shown in Figure 3.13. Notice that the point p on the left hand side of the equation above and the point q on the right hand side are related by the direction l_p of the vector connecting p to q.

While the radiance is used for energy that is emitted, the quantity that describes incoming energy is called *irradiance*. The irradiance is defined as the amount of energy received along a certain direction, per unit area and per unit time. Notice that in the case of the irradiance, we *do not* foreshorten the surface area as in the case of the radiance. Denote the irradiance at p received in the direction l_p by

[8] We recall from the previous chapter that if we represent the change of coordinates g with a rotation matrix $R \in SO(3)$ and a translation vector T, then the action of g on a point p of coordinates $\boldsymbol{X} \in \mathbb{R}^3$ is given by $g(\boldsymbol{X}) \doteq R\boldsymbol{X} + T$, while the action of g on a *vector* of coordinates u is given by $g_*(u) \doteq Ru$.

$dI(p, l_p)$. By energy preservation, we have $dI(p, l_p)\, dA_p\, dt = dE(p, l_p)$. Then the radiance $\mathcal{R}$ at a point q that illuminates the surface dA_p along the direction l_p with a solid angle $d\omega$ and the irradiance dI measured at the same surface dA_p received from this direction are related by

$$dI(p, l_p) = \mathcal{R}(q, l_p) \cos(\theta)\, d\omega, \tag{3.29}$$

where $d\omega = \frac{\cos(\theta_q)}{d(p,q)^2} dA_q$ is the solid angle of dA_q seen from p.

Bidirectional reflectance distribution function

For many common materials, the portion of energy coming from a direction l_p that is reflected onto a direction $\boldsymbol{x}_p$ (i.e. the direction of the vantage point) by the surface S, is described by $\beta(\boldsymbol{x}_p, l_p)$, the *bidirectional reflectance distribution function* (BRDF). Here both $\boldsymbol{x}_p$ and l_p are vectors expressed in local coordinates at p. More precisely, if $d\mathcal{R}(p, \boldsymbol{x}_p, l_p)$ is the radiance emitted in the direction $\boldsymbol{x}_p$ from the irradiance $dI(p, l_p)$, the BRDF is given by the ratio

$$\beta(\boldsymbol{x}_p, l_p) \doteq \frac{d\mathcal{R}(p, \boldsymbol{x}_p, l_p)}{dI(p, l_p)} = \frac{d\mathcal{R}(p, \boldsymbol{x}_p, l_p)}{\mathcal{R}(q, l_p) \cos(\theta)\, d\omega}. \tag{3.30}$$

To obtain the total radiance at a point p in the outgoing direction $\boldsymbol{x}_p$, we need to integrate the BRDF against all the incoming irradiance directions l_p in the hemisphere Ω at p:

$$\mathcal{R}(p, \boldsymbol{x}_p) = \int_\Omega d\mathcal{R}(p, \boldsymbol{x}_p, l_p) = \int_\Omega \beta(\boldsymbol{x}_p, l_p)\, \mathcal{R}(q, l_p) \cos(\theta)\, d\omega. \tag{3.31}$$

Lambertian surfaces

The above model can be considerably simplified if we restrict our attention to a class of materials, called *Lambertian*, that do not change appearance depending on the viewing direction. For example, matte surfaces are to a large extent well approximated by the Lambertian model, since they diffuse light almost uniformly in all directions. Metal, mirrors, and other shiny surfaces, however, do not. Figure 3.14 illustrates a few common surface properties.

For a perfect Lambertian surface, its radiance $\mathcal{R}(p, \boldsymbol{x}_p)$ only depends on how the surface faces the light source, but not on the direction $\boldsymbol{x}_p$ from which it is viewed. Therefore, $\beta(\boldsymbol{x}_p, l_p)$ is actually independent of $\boldsymbol{x}_p$, and we can think of the radiance function as being "glued," or "painted" on the surface S, so that at each point p the radiance $\mathcal{R}$ depends only on the surface. Hence, the perceived irradiance will depend only on which point on the surface is seen, not on in which direction it is seen. More precisely, for Lambertian surfaces, we have

$$\beta(\boldsymbol{x}_p, l_p) = \rho(p),$$

where $\rho(p) : \mathbb{R}^3 \mapsto \mathbb{R}_+$ is a scalar function. In this case, we can easily compute the *surface albedo* ρ_a, which is the percentage of incident irradiance reflected in

Figure 3.14. This figure demonstrates different surface properties widely used in computer graphics to model surfaces of natural objects: Lambertian, diffuse, reflective, specular (highlight), transparent with refraction, and textured. Only the (wood textured) pyramid exhibits Lambertian reflection. The ball on the right is partly ambient, diffuse, reflective and specular. The checkerboard floor is partly ambient, diffuse and reflective. The glass ball on the left is both reflective and refractive.

any direction, as

$$\begin{aligned} \rho_a(p) &= \int_\Omega \beta(\boldsymbol{x}_p, l_p)\cos(\theta_p)\, d\omega_p = \rho(p)\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \cos(\theta_p)\sin(\theta_p)\, d\theta_p\, d\phi_p \\ &= \pi\rho(p), \end{aligned}$$

where $d\omega_p$, as shown in Figure 3.13, is the infinitesimal solid angle in the outgoing direction, which can be parameterized by the space angles (θ_p, ϕ_p) as $d\omega_p = \sin(\theta_p)d\theta_p d\phi_p$. Hence the radiance from the point p on a Lambertian surface S is

$$\mathcal{R}(p) = \int_\Omega \frac{1}{\pi}\rho_a(p)\, \mathcal{R}(q, l_p)\cos(\theta)\, d\omega. \tag{3.32}$$

This equation is known as *Lambertian cosine law*. Therefore, for a Lambertian surface, the radiance $\mathcal{R}$ depends only on the surface S, described by its generic point p, and on the light source L, described by its radiance $\mathcal{R}(q, l_p)$.

Image intensity for a Lambertian surface

In order to express the direction $\boldsymbol{x}_p$ in the camera frame, we consider the change of coordinates from the local coordinate frame at the point p to the camera frame: $\boldsymbol{X}(p) \doteq g(0)$ and $\boldsymbol{x} \sim g_*(\boldsymbol{x}_p)$, where we note that g_* is a rotation.[9] The reader should be aware that the transformation g itself depends on local shape of the

[9] The symbol $\sim$ indicates equivalence up to a scalar factor. Strictly speaking, $\boldsymbol{x}$ and $g_*(\boldsymbol{x}_p)$ do not represent the same vector, but only the same direction (they have opposite sign and different lengths). To obtain a rigorous expression, we would have to write $\boldsymbol{x} = \pi(-g_*(\boldsymbol{x}_p))$. However, these

surface at p, in particular its tangent plane T_pS and its normal ν_p at the point p. We now can rewrite the expression (3.31) for the radiance in terms of the camera coordinates and obtain

$$\mathcal{R}(\boldsymbol{X}) \doteq \mathcal{R}(p, g_*^{-1}(\boldsymbol{x})), \quad \text{where} \quad \boldsymbol{x} = \pi(\boldsymbol{X}). \tag{3.33}$$

If the surface is Lambertian, the above expression simplifies to

$$\mathcal{R}(\boldsymbol{X}) = \mathcal{R}(p). \tag{3.34}$$

Suppose that our imaging sensor is well modeled by a thin lens. Then, by measuring the amount of energy received along the direction $\boldsymbol{x}$, the irradiance (or image intensity) I at $\boldsymbol{x}$ can be expressed as a function of the radiance from the point p:

$$I(\boldsymbol{x}) = \mathcal{R}(\boldsymbol{X})\frac{\pi}{4}\left(\frac{d}{f}\right)^2 \cos^4(\alpha), \tag{3.35}$$

where d is the lens diameter, f is the focal length, and α is the angle between the optical axis (i.e. the z-axis) and the image point $\boldsymbol{x}$, as shown in Figure 3.13. The quantity $\frac{d}{f}$ is called the *F-number* of the lens. A detailed derivation of the above formula can be found in [Horn, 1986] (page 208). For a Lambertian surface, we have

$$\begin{aligned} I(\boldsymbol{x}) &= \mathcal{R}(\boldsymbol{X})\frac{\pi}{4}\left(\frac{d}{f}\right)^2 \cos^4(\alpha) = \mathcal{R}(p)\frac{\pi}{4}\left(\frac{d}{f}\right)^2 \cos^4(\alpha) \\ &= \frac{1}{4}\left(\frac{d}{f}\right)^2 \cos^4(\alpha) \int_\Omega \rho_a(p)\, \mathcal{R}(q, l_p) \cos(\theta)\, d\omega, \end{aligned}$$

where $\boldsymbol{x}$ is the image of the point p taken at the vantage point g. Notice that in the above expression, only the angle α depends on the vantage point. In general, for a thin lens with a small field of view, α is approximately constant. Therefore, in our ideal pin-hole model, we may assume that the image intensity (i.e. irradiance) is related to the surface radiance by the *irradiance equation*:

$$\boxed{I(\boldsymbol{x}) = \gamma \mathcal{R}(p),} \tag{3.36}$$

where $\gamma \doteq \frac{\pi}{4}\left(\frac{d}{f}\right)^2 \cos^4(\alpha)$ is a constant factor that is independent of the vantage point.

In all subsequent chapters we will adopt this simple model. The fact that the irradiance I does not change with the vantage point for Lambertian surfaces constitutes a fundamental condition that allows to establish correspondence across multiple images of the same object. This condition and its implications will be studied in more detail in the next chapter.

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two vectors do represent the same ray through the camera center, and therefore we will regard them as the same.