Segmentation of Subspace Arrangements III – Robust GPCA

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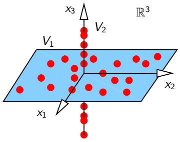
Generalized Principal Component Analysis (GPCA): (an overview)



- $\mathbf{x} \in V_1 \cup V_2 \Rightarrow (x_3 = 0) or (x_1 = x_2 = 0)$
  $\Rightarrow \{x_1 x_3 = 0, x_2 x_3 = 0\}$.
- *Veronese Map*: Given N samples $\mathbf{x}_1, \dots, \mathbf{x}_N \in \mathbb{R}^3$,

$$L_2 \doteq [\nu_2(\mathbf{x}_1), \dots, \nu_2(\mathbf{x}_N)] \in \mathbb{R}^{M_2^{[3]} \times N} = \begin{bmatrix} \cdots & (x_1)^2 & \cdots \\ \cdots & (x_1 x_2) & \cdots \\ \cdots & (x_1 x_3) & \cdots \\ \cdots & (x_2)^2 & \cdots \\ \cdots & (x_2 x_3) & \cdots \\ \cdots & (x_3)^2 & \cdots \end{bmatrix}$$

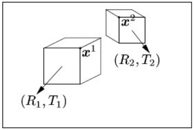
- The null space of L_2 is $\begin{matrix} \mathbf{c}_1 = [0,0,1,0,0,0] \\ \mathbf{c}_2 = [0,0,0,0,1,0] \end{matrix} \Rightarrow \begin{matrix} p_1 = \mathbf{c}_1 \nu_2(\mathbf{x}) = x_1 x_3 \\ p_2 = \mathbf{c}_2 \nu_2(\mathbf{x}) = x_2 x_3 \end{matrix}$
- $P(\mathbf{x}) \doteq [p_1(\mathbf{x})\ p_2(\mathbf{x})] = [x_1 x_3, x_2 x_3]$, then
  $\nabla_{\mathbf{x}} P = [\, \nabla_{\mathbf{x}} p_1 \ \nabla_{\mathbf{x}} p_2 \,] = \begin{bmatrix} x_3 & 0 \\ 0 & x_3 \\ x_1 & x_2 \end{bmatrix}$.
- $\nabla_{\mathbf{x}} P$ at one sample per subspace gives normal vectors that span $V_1^\perp$ and $V_2^\perp$:

$$\mathbf{x} = [a, b, 0]^T \in V_1 \Rightarrow \nabla_{\mathbf{x}} P|_{\mathbf{x}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ a & b \end{bmatrix} \in V_1^\perp.$$

$$\mathbf{y} = [0, 0, c]^T \in V_2 \Rightarrow \nabla_{\mathbf{x}} P|_{\mathbf{y}} = \begin{bmatrix} c & 0 \\ 0 & c \\ 0 & 0 \end{bmatrix} \in V_2^\perp.$$

- Segment samples and recover V_1 and V_2.

Multibody Fundamental Matrix



- Given K rigid bodies, an image correspondence $(\mathbf{x}_1, \mathbf{x}_2)$ satisfies:

$$f(\mathbf{x}_1, \mathbf{x}_2) = (\mathbf{x}_2^T F_1 \mathbf{x}_1)(\mathbf{x}_2^T F_2 \mathbf{x}_1) \cdots (\mathbf{x}_2^T F_K \mathbf{x}_1) = 0.$$

Using the kronecker product:

$$f(\mathbf{x}_1, \mathbf{x}_2) = \{(\mathbf{x}_1 \otimes \mathbf{x}_2)^T F_1^s\} \cdots \{(\mathbf{x}_1 \otimes \mathbf{x}_2)^T F_K^s\}.$$

We treat $\mathbf{z} = \mathbf{x}_1 \otimes \mathbf{x}_2 \in \mathbb{R}^9$ as the new feature vector, then

$$f(\mathbf{x}_1, \mathbf{x}_2) = \nu_K(\mathbf{z})^T \mathbf{c}.$$

- A second way to rewrite the bilinear constraint:
  - $f(\mathbf{x}_1, \mathbf{x}_2)$ is a linear constraint with respect to $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively:

$$f(\mathbf{x}_1, \mathbf{x}_2) = f_{\mathbf{x}_2}(\mathbf{x}_1) = \mathbf{c}_{\mathbf{x}_2}^T \nu_K(\mathbf{x}_1); \quad f(\mathbf{x}_1, \mathbf{x}_2) = f_{\mathbf{x}_1}(\mathbf{x}_2) = \mathbf{c}_{\mathbf{x}_1}^T \nu_K(\mathbf{x}_2).$$

  - Hence, $f(\mathbf{x}_1, \mathbf{x}_2)$ can be rewritten by applying the Veronese map individually:

$$f(\mathbf{x}_1, \mathbf{x}_2) = \nu_K(\mathbf{x}_2)^T \mathcal{F} \nu_K(\mathbf{x}_1), \text{ where } \mathcal{F} \in \mathbb{R}^{M_K^{[3]} \times M_K^{[3]}}.$$

- Which representation is more compact?
  1. $K = 2$: For $\mathbf{c}$, $M_2^{[9]} = \binom{2+9-1}{2} = 45$. For $\mathcal{F}$, $(M_2^{[3]})^2 = 6^2 = 36$.
  2. $K = 4$: For $\mathbf{c}$, $M_4^{[9]} = 495$. For $\mathcal{F}$, $(M_4^{[3]})^2 = 15^2 = 225$.

  Choose the second one.
- $\mathcal{F}$ is called the multibody fundamental matrix, comparing to the (single) fundamental matrix F.

Segmentation of Multibody Motion

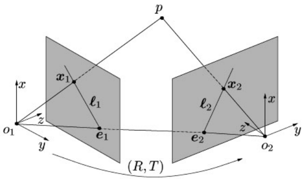
- The vanishing polynomial $f(\mathbf{x}_1, \mathbf{x}_2)$ is solved as the following:

$$f(\mathbf{x}_1, \mathbf{x}_2) = \nu_K(\mathbf{x}_2)^T \mathcal{F} \nu_K(\mathbf{x}_1) = (\nu_K(\mathbf{x}_1) \otimes \nu_K(\mathbf{x}_2))^T \mathcal{F}^s.$$

- $\frac{\partial}{\partial \mathbf{x}_2} f(\mathbf{x}_1, \mathbf{x}_2) = \sum_{i=1}^K \left(\prod_{j \neq i} \mathbf{x}_2^T F_j \mathbf{x}_1\right)(F_i \mathbf{x}_1)$.
- Suppose $(\mathbf{x}_1, \mathbf{x}_2)$ on Object k, then $\left(\prod_{j \neq i} \mathbf{x}_2^T F_j \mathbf{x}_1\right) = 0$ for all $i \neq k$:

$$\frac{\partial}{\partial \mathbf{x}_2} f(\mathbf{x}_1, \mathbf{x}_2) = \left(\prod_{j \neq k} \mathbf{x}_2^T F_j \mathbf{x}_1\right)(F_k \mathbf{x}_1) \sim (F_k \mathbf{x}_1) \sim \mathbf{l}_k^2.$$

Hence, $\frac{\partial}{\partial \mathbf{x}_2} f(\mathbf{x}_1, \mathbf{x}_2) \sim \mathbf{l}_k^2 \perp \mathbf{e}_k^2$.



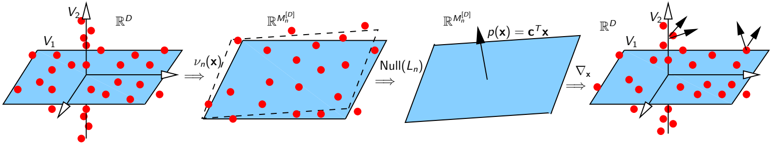
- Given K rigid body motions, there exist K different epipoles $\mathbf{e}_1, \cdots, \mathbf{e}_K$ in the second view such that:

$$(\mathbf{e}_1^T \mathbf{l})(\mathbf{e}_2^T \mathbf{l}) \cdots (\mathbf{e}_K^T \mathbf{l}) = 0,$$

which is a standard subspace-segmentation problem.

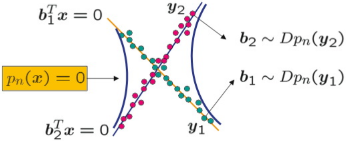
GPCA-Voting: A Stable Implementation

PDA on noisy data



The noise affects the algebraic PDA process:

1. The data matrix $L_K(V)$ is always *full-rank*.
   Solution: Use SVD to estimate Null(L_K).
2. How to choose one point per subspace as the representative?
   Solution: Rule of thumb is to pick samples far away from the origin and intersections.



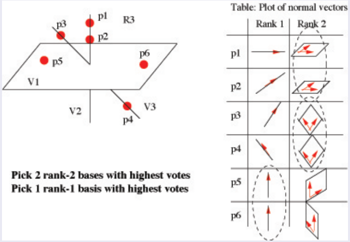
3. Even with a good sample, evaluation of ∇P is still perturbed away from the true position.
   Solution: We propose a voting algorithm to evaluate ∇P at all samples.

A Voting Scheme

- Goal:
  1. Averaging $\nabla_{\mathbf{x}} P$ at more samples of a subspace.
  2. Recover correct rank of $\nabla_{\mathbf{x}} P$.
- Difficulty: Do not know which samples belong to the same subspace, yet.

GPCA-Voting (a simple example)

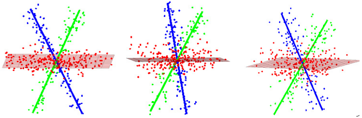
1. Assume subspaces $(2, 1, 1)$ in $\mathbb{R}^3$.
2. $h_I(3) = 4$ vanishing polynomials $\Rightarrow \nabla_{\mathbf{x}} P \in \mathbb{R}^{3\times 4}$.
3. Vote on rank-1 & rank-2 codimensions with a tolerance threshold τ



4. Average normal vectors associated with highest votes.
5. (optional) Iteratively refine the segmentation via EM or K-Subspaces.

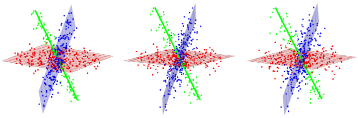
Simulation Results

1. Illustrations



(a) 8% (b) 12% (c) 16%

Figure: $(2,1,1) \in \mathbb{R}^3$.



(a) 8% (b) 12% (c) 16%

Figure: $(2,2,1) \in \mathbb{R}^3$.

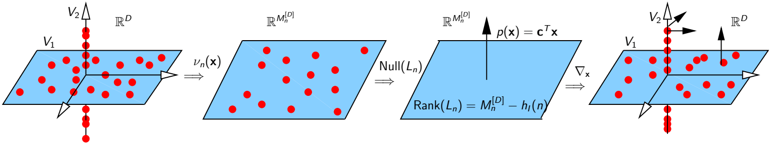
2. Segmentation simulations

Table: Segmentation errors. 4% Gaussian noise is added.

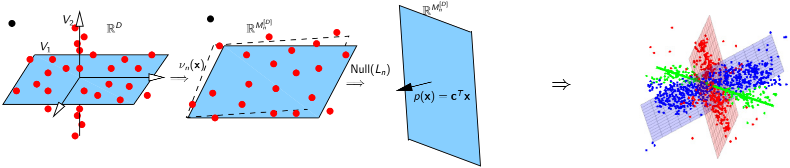
Subspace Dimensions	EM	K-Subspaces	PDA	Voting	Voting+K-Subspaces
$(2,2,1)$ in $\mathbb{R}^3$	29%	27%	13.2%	6.4%	5.4%
$(4,2,2,1)$ in $\mathbb{R}^5$	53%	57%	39.8%	5.7%	5.7%
$(4,4,4,4)$ in $\mathbb{R}^5$	20%	25%	25.3%	17%	11%

Outlier Issue

- GPCA process:



- Breakdown of GPCA is 0% because breakdown of PCA is 0%: a large outlier can arbitrarily perturb $\mathrm{Null}(L_n)$

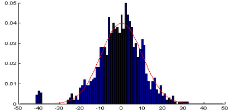


$\Rightarrow$ Seek a robust PCA to estimate $\mathrm{Null}(L_n)$, where $L_n = [\nu_n(\mathbf{x}_1), \cdots, \nu_n(\mathbf{x}_N)]$.

Three approaches to tackle outliers:

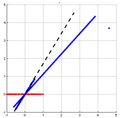
1. Probability-based: small-probability samples.

   Probability plots: [Healy 1968, Cox 1968]
   PCs: [Rao 1964, Ganadesikan & Kettenring 1972]
   M-estimators: [Huber 1981, Campbell 1980]
   multivariate trimming (MVT): [Ganadesikan & Kettenring 1972]



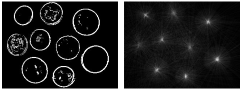
2. Influence-based: large influence on model parameters.

   Parameter difference with and without a sample: [Hampel et al. 1986, Critchley 1985]



3. Consensus-based: not consistent with models of high consensus.

   Hough: [Ballard 1981, Lowe 1999]
   RANSAC: [Fischler & Bolles 1981, Torr 1997]
   Least Median Estimate (LME): [Rousseeuw 1984, Steward 1999]

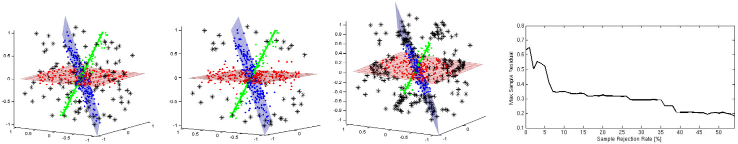


Robust GPCA

STEP 1: Given the outlier percentage $\alpha\%$, robustify PCA:

- Influence function:
  1. Compute null space $C = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_m\}$ for $L_n = [\nu_n(\mathbf{x}_1) \cdots \nu_n(\mathbf{x}_N)]$.
  2. For $\mathbf{x}_i$, compute $C^{(i)}$ for $L_n^{(i)} = [\nu_n(\mathbf{x}_1) \cdots \hat{i} \cdots \nu_n(\mathbf{x}_N)]$.
  3. $I(\mathbf{x}_i) \doteq \langle C, C^{(i)} \rangle$.
  4. Reject top $\alpha\%$ samples with highest influence.
- Multivariate-trimming (MVT):
  Assuming a Gaussian distribution, samples with large *Mahalanobis* distance more likely to be outliers.
  1. Compute a robust mean $\bar{u}$. $\mathbf{v}_i = \mathbf{u}_i - \bar{u}$. $\mathbf{u}_i, \mathbf{v}_i \in \mathbb{R}^{M_n^{[D]}}$
  2. Initialize $\Sigma_0 = I_{M_n^{[D]} \times M_n^{[D]}}$.
  3. In kth iteration, sort $\mathbf{v}_1, \ldots, \mathbf{v}_N$ by the *Mahalanobis* distance:
  $$d_i = \mathbf{v}_i^T \Sigma_{k-1}^{-1} \mathbf{v}_i.$$
  4. Update Σ_k from $(100 - \alpha)\%$ samples with smallest distances.
  5. Iteration stops when $\|\Sigma_{k-1} - \Sigma_k\|$ is small.

STEP 2: Estimating the outlier percentage $\alpha\%$: Do we need the exact percentage for estimation?



(a) 16% outliers

(b) 7% rejected

(c) 38% rejected

(d) Maximal sample residuals.

1. With noise and outliers present, unnecessary to distinguish outliers close to subspaces.
2. Robust PCA is moderately stable when the outlier percentage is over-estimated.

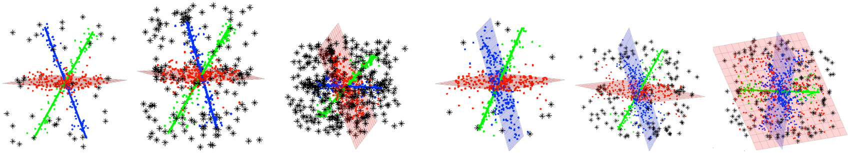
Outlier Percentage Test based on the Influence Function Principle

Further rejection only results in small changes in the model parameters and sample residuals (w.r.t. boundary threshold σ), i.e., the arrangement model stabilizes.

Influence

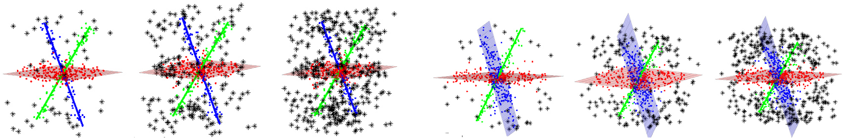
Simulations on Robust GPCA (parameters fixed at $\tau = 0.3$rad and $\sigma = 0.4$)

- RGPCA-Influence



(e) 12% (f) 32% (g) 48% (h) 12% (i) 32% (j) 48%

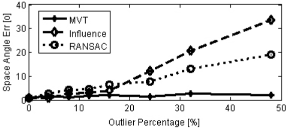
- RGPCA-MVT



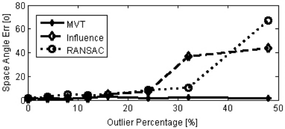
(k) 12% (l) 32% (m) 48% (n) 12% (o) 32% (p) 48%

Comparison with RANSAC

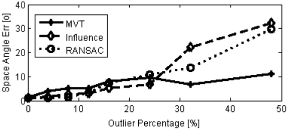
- Accuracy



(q) $(2, 2, 1)$ in $\mathbb{R}^3$



(r) $(4, 2, 2, 1)$ in $\mathbb{R}^5$



(s) $(5, 5, 5)$ in $\mathbb{R}^6$

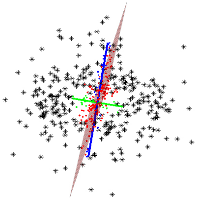
- Speed

Table: Average time of RANSAC and RGPCA with 24% outliers.

Arrangement	(2,2,1) in $\mathbb{R}^3$	(4,2,2,1) in $\mathbb{R}^5$	(5,5,5) in $\mathbb{R}^6$
RANSAC	44s	5.1m	3.4m
MVT	46s	23m	8m
Influence	3m	58m	146m

Limitations of RGPCA

1. Hardware limits for high subspace dimension (> 10) or subspace number (> 6) in MATLAB.
2. Need to know the number of subspaces and dimensions.
3. Overfitting when percentage is overestimated, especially for MVT.

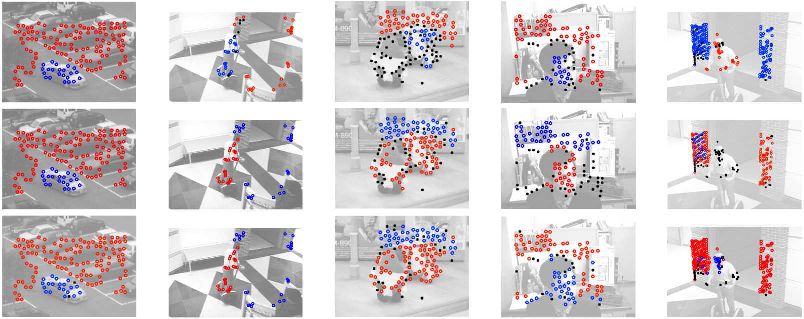
$\Longrightarrow$ 

Animation

Next section, we show solutions to these limitations via a new lossy coding framework.

Experiment 1: Motion Segmentation under 3-D Affine Projection

Sequences: parking-lot | segway | toys | man | segway3



RANSAC:

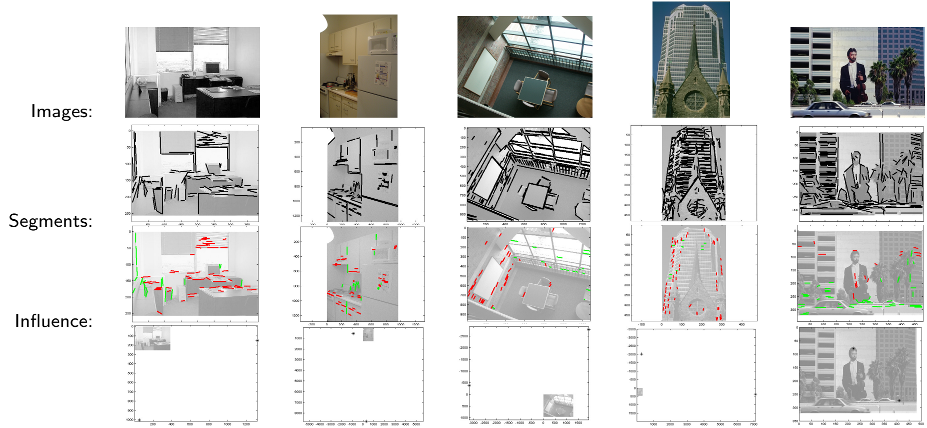
MVT:

Influence:

Feature extraction: Shi and Tomasi. **Good features to track**. *CVPR* 1994.

Experiment 2: Vanishing-Point Detection

RGPCA-Influence



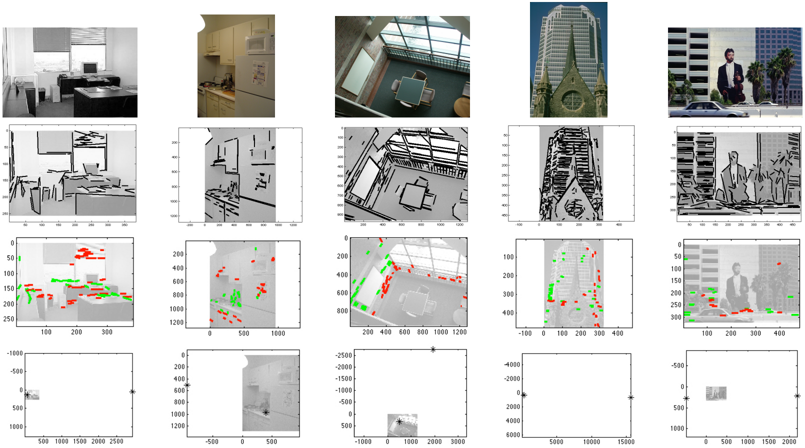
Feature extraction: Kahn et al. **A fast line finder for vision-guided robot navigation**. *PAMI*, 1990.

RGPCA-MVT

Images:

Segments:

MVT:

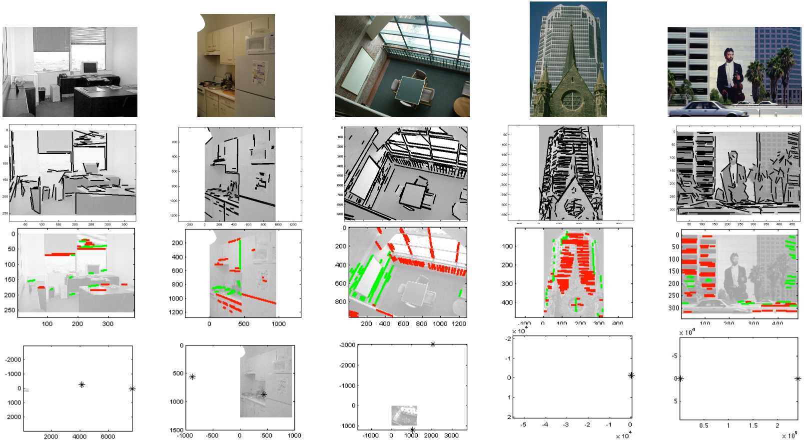


RANSAC-on-Subspaces

Images:

Segments:

RANSAC:



Conclusions

- Estimation of hybrid subspace models is closely related to the study of subspace arrangements in algebraic geometry.
- Global structure of K subspaces uniquely determined by Kth degree vanishing polynomials.
- Two algorithms were proposed using vanishing polynomials as a global signature
  1. Noise: GPCA-Voting .
  2. Outliers: RGPCA.

Confluence of Algebra and Statistics

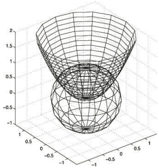
In estimation of hybrid subspace models:

- Algebra makes statistical algorithms well-conditioned;
- Statistics makes algebraic algorithms robust.

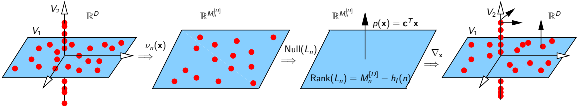
Future Directions

- Mathematics: More complex models [Rao et al. *ICCV* 2005].

  1. A union of quadratic surfaces.
  2. A mixture of linear subspaces and quadratic surfaces.



- Kernel GPCA: tackle the curse of dimensionality in Veronese embedding.



- Model selection: a unified scheme to tackle simultaneous model estimation and selection.
- Applications

  1. Natural image compression and classification.
  2. Hyper-spectral image analysis.