Segmentation of Subspace Arrangements III – Robust GPCA

Allen Y. Yang

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Generalized Principal Component Analysis (GPCA): (an overview)

- $\mathbf{x} \in V_1 \cup V_2 \Rightarrow (x_3 = 0) \text{or} (x_1 = x_2 = 0) \\
  \Rightarrow \{x_1x_3 = 0, x_2x_3 = 0\}$.

- **Veronese Map**: Given $N$ samples $\mathbf{x}_1, \ldots, \mathbf{x}_N \in \mathbb{R}^3$,
  
  $$L_2 \doteq [\nu_2(\mathbf{x}_1), \ldots, \nu_2(\mathbf{x}_N)] \in \mathbb{R}^{M_2^3} \times N$$

  
  $$= \begin{bmatrix}
  \cdots (x_1)^2 & \cdots \\
  \cdots (x_1x_2) & \cdots \\
  \cdots (x_1x_3) & \cdots \\
  \cdots (x_2)^2 & \cdots \\
  \cdots (x_2x_3) & \cdots \\
  \cdots (x_3)^2 & \cdots 
  \end{bmatrix}$$

  
  The null space of $L_2$ is $\mathbf{c}_1 = [0, 0, 1, 0, 0, 0] \Rightarrow p_1 = \mathbf{c}_1 \nu_2(\mathbf{x}) = x_1x_3$

  $\mathbf{c}_2 = [0, 0, 0, 0, 1, 0] \Rightarrow p_2 = \mathbf{c}_2 \nu_2(\mathbf{x}) = x_2x_3$

  $P(\mathbf{x}) \doteq [p_1(\mathbf{x}) \ p_2(\mathbf{x})] = [x_1x_3, x_2x_3]$, then

  $$\nabla_{\mathbf{x}} P = \begin{bmatrix}
  \nabla_{x_1p_1} & \nabla_{x_1p_2} \\
  \nabla_{x_2p_1} & \nabla_{x_2p_2}
  \end{bmatrix} = \begin{bmatrix}
  x_3 & 0 \\
  0 & x_3 \\
  x_1 & x_2
  \end{bmatrix}.$$ 

  $\nabla_{\mathbf{x}} P$ at one sample per subspace gives normal vectors that span $V_1^\perp$ and $V_2^\perp$:

  - $\mathbf{x} = [a, b, 0]^T \in V_1 \Rightarrow \nabla_{\mathbf{x}} P|_{\mathbf{x}} = \begin{bmatrix}
  0 & 0 \\
  0 & a \\
  0 & b
  \end{bmatrix} \in V_1^\perp$.

  - $\mathbf{y} = [0, 0, c]^T \in V_2 \Rightarrow \nabla_{\mathbf{x}} P|_{\mathbf{y}} = \begin{bmatrix}
  c & 0 \\
  0 & c \\
  0 & 0
  \end{bmatrix} \in V_2^\perp$.

- Segment samples and recover $V_1$ and $V_2$. 

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Segmentation of Subspace Arrangements III – Robust GPCA
1 Multibody Epipolar Constraint
   • Multibody Fundamental Matrix

2 GPCA-Voting
   • Noise issue
   • GPCA-Voting
   • Comparison

3 Robust GPCA
   • Outlier Issue
   • Robustifying GPCA via Influence and MVT
   • Comparison

4 Applications
   • Affine Motion Detection
   • Vanishing-Point Detection

5 Conclusion and Discussion
   • Conclusion
   • Future Directions
Multibody Fundamental Matrix

Given $K$ rigid bodies, an image correspondence $(x_1, x_2)$ satisfies:

$$f(x_1, x_2) = (x_2^T F_1 x_1)(x_2^T F_2 x_1) \cdots (x_2^T F_K x_1) = 0.$$  

Using the kronecker product:

$$f(x_1, x_2) = \{(x_1 \otimes x_2)^T F_1^s\} \cdots \{(x_1 \otimes x_2)^T F_K^s\}.$$  

We treat $z = x_1 \otimes x_2 \in \mathbb{R}^9$ as the new feature vector, then

$$f(x_1, x_2) = \nu_K(z)^T c.$$  

A second way to rewrite the bilinear constraint:

- $f(x_1, x_2)$ is a linear constraint with respect to $x_1$ and $x_2$, respectively:

  $$\begin{align*}
  f(x_1, x_2) &= f_{x_2}(x_1) = c_{x_2}^T \nu_K(x_1); \\
  f(x_1, x_2) &= f_{x_1}(x_2) = c_{x_1}^T \nu_K(x_2).
  \end{align*}$$  

- Hence, $f(x_1, x_2)$ can be rewritten by applying the Veronese map individually:

  $$f(x_1, x_2) = \nu_K(x_2)^T \mathcal{F} \nu_K(x_1), \quad \text{where} \quad \mathcal{F} \in \mathbb{R}^{M_K^3 \times M_K^3}.$$  

Which representation is more compact?

1. $K = 2$: For $c$, $M_2^9 = \binom{2+9-1}{2} = 45$. For $\mathcal{F}$, $(M_2^3)^2 = 6^2 = 36$.
2. $K = 4$: For $c$, $M_4^9 = 495$. For $\mathcal{F}$, $(M_4^3)^2 = 15^2 = 225$.

Choose the second one.

- $\mathcal{F}$ is called the multibody fundamental matrix, comparing to the (single) fundamental matrix $F$.  

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Segmentation of Multibody Motion

- The vanishing polynomial \( f(x_1, x_2) \) is solved as the following:

\[
  f(x_1, x_2) = \nu_K(x_2)^T \mathcal{F} \nu_K(x_1) = (\nu_K(x_1) \otimes \nu_K(x_2))^T \mathcal{F}^s.
\]

- \[
  \frac{\partial}{\partial x_2} f(x_1, x_2) = \sum_{i=1}^{K} \left( \prod_{j \neq i} x_2^T F_j x_1 \right) (F_i x_1).
\]

- Suppose \((x_1, x_2)\) on Object \(k\), then \( \left( \prod_{j \neq i} x_2^T F_j x_1 \right) = 0 \) for all \(i \neq k\):

\[
  \frac{\partial}{\partial x_2} f(x_1, x_2) = \left( \prod_{j \neq k} x_2^T F_j x_1 \right) (F_k x_1) \sim (F_k x_1) \sim I_k^2.
\]

Hence, \( \frac{\partial}{\partial x_2} f(x_1, x_2) \sim I_k^2 \perp e_k^2 \).

- Given \(K\) rigid body motions, there exist \(K\) different epipoles \(e_1, \cdots, e_K\) in the second view such that:

\[
  (e_1^T l)(e_2^T l) \cdots (e_K^T l) = 0,
\]

which is a standard subspace-segmentation problem.
GPCA-Voting: A Stable Implementation

PDA on noisy data

The noise affects the algebraic PDA process:

1. The data matrix $L_K(V)$ is always full-rank.  
   **Solution:** Use SVD to estimate $\text{Null}(L_K)$.

2. How to choose one point per subspace as the representative?  
   **Solution:** Rule of thumb is to pick samples far away from the origin and intersections.

3. Even with a good sample, evaluation of $\nabla P$ is still perturbed away from the true position.  
   **Solution:** We propose a voting algorithm to evaluate $\nabla P$ at all samples.

\[ b_1^T x = 0 \quad \Rightarrow \quad p_n(x) = 0 \]
\[ b_2^T x = 0 \quad \Rightarrow \quad b_2 \sim Dp_n(y_2) \]
\[ b_1 \sim Dp_n(y_1) \]
A Voting Scheme

- **Goal:**
  1. Averaging $\nabla_x P$ at more samples of a subspace.
  2. Recover correct rank of $\nabla_x P$.

- **Difficulty:** Do not know which samples belong to the same subspace, yet.

**GPCA-Voting (a simple example)**

1. Assume subspaces $(2, 1, 1)$ in $\mathbb{R}^3$.
2. $h_I(3) = 4$ vanishing polynomials $\Rightarrow \nabla_x P \in \mathbb{R}^{3 \times 4}$.
3. Vote on rank-1 & rank-2 codimensions with a tolerance threshold $\tau$.
4. Average normal vectors associated with highest votes.
5. (optional) Iteratively refine the segmentation via EM or K-Subspaces.
Simulation Results

1. Illustrations

![Illustrations](image.png)

(a) 8%  
(b) 12%  
(c) 16%

Figure: $(2, 1, 1) \in \mathbb{R}^3$.

(a) 8%  
(b) 12%  
(c) 16%

Figure: $(2, 2, 1) \in \mathbb{R}^3$.

2. Segmentation simulations

Table: Segmentation errors. 4% Gaussian noise is added.

<table>
<thead>
<tr>
<th>Subspace Dimensions</th>
<th>EM</th>
<th>K-Subspaces</th>
<th>PDA</th>
<th>Voting</th>
<th>Voting+K-Subspaces</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2, 2, 1) \in \mathbb{R}^3$</td>
<td>29%</td>
<td>27%</td>
<td>13.2%</td>
<td>6.4%</td>
<td>5.4%</td>
</tr>
<tr>
<td>$(4, 2, 2, 1) \in \mathbb{R}^5$</td>
<td>53%</td>
<td>57%</td>
<td>39.8%</td>
<td>5.7%</td>
<td>5.7%</td>
</tr>
<tr>
<td>$(4, 4, 4, 4) \in \mathbb{R}^5$</td>
<td>20%</td>
<td>25%</td>
<td>25.3%</td>
<td>17%</td>
<td>11%</td>
</tr>
</tbody>
</table>
Outlier Issue

GPCA process:

$$\begin{align*}
V_1 & \rightarrow R^D \\
n(x) & \Rightarrow Null(L_n) \\
\text{Rank}(L_n) & = M_n^{[D]} - h_l(n) \\
R & \Rightarrow V_2 \rightarrow R^D \\
\end{align*}$$

Breakdown of GPCA is 0% because breakdown of PCA is 0%: a large outlier can arbitrarily perturb $Null(L_n)$

$$\begin{align*}
V_1 & \rightarrow R^D \\
n(x) & \Rightarrow Null(L_n) \\
p(x) & = c^T x \\
\Rightarrow & \\
\end{align*}$$

⇒ Seek a robust PCA to estimate $Null(L_n)$, where $L_n = [n_1(x_1), \cdots, n_n(x_N)]$. 
Three approaches to tackle outliers:

1. **Probability-based**: small-probability samples.
   
   *Probability plots*: [Healy 1968, Cox 1968]  
   *PCs*: [Rao 1964, Ganadesikan & Kettenring 1972]  
   *M-estimators*: [Huber 1981, Campbell 1980]  
   *Multivariate trimming* (MVT): [Ganadesikan & Kettenring 1972]

2. **Influence-based**: large influence on model parameters.
   
   Parameter difference with and without a sample: [Hampel et al. 1986, Critchley 1985]

3. **Consensus-based**: not consistent with models of high consensus.
   
Robust GPCA

STEP 1: Given the outlier percentage $\alpha\%$, robustify PCA:

- **Influence function:**
  1. Compute null space $C = \{c_1, c_2, \ldots, c_m\}$ for $L_n = [\nu_n(x_1) \cdots \nu_n(x_N)]$.
  2. For $x_i$, compute $C^{(i)}$ for $L_n^{(i)} = [\nu_n(x_1) \cdots \hat{i} \cdots \nu_n(x_N)]$.
  3. $I(x_i) = \langle C, C^{(i)} \rangle$.
  4. Reject top $\alpha\%$ samples with highest influence.

- **Multivariate-trimming (MVT):**
  Assuming a Gaussian distribution, samples with large Mahalanobis distance more likely to be outliers.
  1. Compute a robust mean $\bar{u}$. $v_i = u_i - \bar{u}$, $u_i, v_i \in \mathbb{R}^{M[D]}$
  2. Initialize $\Sigma_0 = I_{M_n[D] \times M_n[D]}$.
  3. In $k$th iteration, sort $v_1, \ldots, v_N$ by the Mahalanobis distance:
    
    \[ d_i = v_i^T \Sigma_k^{-1} v_i. \]
  4. Update $\Sigma_k$ from $(100 - \alpha\%)$ samples with smallest distances.
  5. Iteration stops when $\|\Sigma_{k-1} - \Sigma_k\|$ is small.
STEP 2: Estimating the outlier percentage $\alpha\%$: Do we need the exact percentage for estimation?

1. With noise and outliers present, unnecessary to distinguish outliers close to subspaces.
2. Robust PCA is moderately stable when the outlier percentage is over-estimated.

**Outlier Percentage Test based on the Influence Function Principle**

Further rejection only results in small changes in the model parameters and sample residuals (w.r.t. boundary threshold $\sigma$), i.e., the arrangement model stabilizes.
Simulations on Robust GPCA (parameters fixed at $\tau = 0.3$rad and $\sigma = 0.4$)

- **RGPCA-Influence**

  - (e) 12%
  - (f) 32%
  - (g) 48%
  - (h) 12%
  - (i) 32%
  - (j) 48%

- **RGPCA-MVT**

  - (k) 12%
  - (l) 32%
  - (m) 48%
  - (n) 12%
  - (o) 32%
  - (p) 48%
Comparison with RANSAC

- **Accuracy**

  ![Graphs showing Accuracy Comparison]

  (q) \((2, 2, 1)\) in \(\mathbb{R}^3\)

  (r) \((4, 2, 2, 1)\) in \(\mathbb{R}^5\)

  (s) \((5, 5, 5)\) in \(\mathbb{R}^6\)

- **Speed**

  **Table**: Average time of RANSAC and RGPCA with 24% outliers.

<table>
<thead>
<tr>
<th>Arrangement</th>
<th>((2,2,1)) in (\mathbb{R}^3)</th>
<th>((4,2,2,1)) in (\mathbb{R}^5)</th>
<th>((5,5,5)) in (\mathbb{R}^6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>RANSAC</td>
<td>44s</td>
<td>5.1m</td>
<td>3.4m</td>
</tr>
<tr>
<td>MVT</td>
<td>46s</td>
<td>23m</td>
<td>8m</td>
</tr>
<tr>
<td>Influence</td>
<td>3m</td>
<td>58m</td>
<td>146m</td>
</tr>
</tbody>
</table>
Limitations of RGPCA

1. Hardware limits for high subspace dimension (> 10) or subspace number (> 6) in MATLAB.
2. Need to know the number of subspaces and dimensions.
3. Overfitting when percentage is overestimated, especially for MVT.

Next section, we show solutions to these limitations via a new lossy coding framework.
Experiment 1: Motion Segmentation under 3-D Affine Projection

Sequences: parking-lot, segway, toys, man, segway3

RANSAC:

MVT:

Influence:

Feature extraction: Shi and Tomasi. **Good features to track.** CVPR 1994.
Experiment 2: Vanishing-Point Detection

RGPCA-Influence

Images:

Segments:

Influence:

RGPCA-MVT

Images:

Segments:

MVT:
RANSAC-on-Subspaces

Images:

Segments:

RANSAC:
Conclusions

- Estimation of hybrid subspace models is closely related to the study of subspace arrangements in algebraic geometry.
- Global structure of $K$ subspaces uniquely determined by $K$th degree vanishing polynomials.
- Two algorithms were proposed using vanishing polynomials as a global signature:
  2. Outliers: RGPCA.

Confluence of Algebra and Statistics

In estimation of hybrid subspace models:
- Algebra makes statistical algorithms well-conditioned;
- Statistics makes algebraic algorithms robust.
Future Directions

• Mathematics: More complex models [Rao et al. ICCV 2005].

1. A union of quadratic surfaces.
2. A mixture of linear subspaces and quadratic surfaces.

• Kernel GPCA: tackle the **curse of dimensionality** in Veronese embedding.

• Model selection: a unified scheme to tackle simultaneous model estimation and selection.

• Applications
  1. Natural image compression and classification.
  2. Hyper-spectral image analysis.